17 Some possible answers to the "symmetry and special cases" section

1. MAT 2011 b

   Suppose it's a 1×1 square. Which of the inequalities is valid? Only (a) and (c). Suppose it is a very tiny square (say side 0.0001). Which of the inequalities is still valid? Only (c).

2. MAT 2014

   The inequality has only even powers of x in it, so it's true for x if it's true for $-x$. Therefore the interval for which it is satisfied must be: symmetrical around zero. Therefore the answer is: (a) $-3 < x < 3$.

3. MAT 2013 d

   Symmetry: rewriting the equation as $x^4 = y^2 + 2y + 1 = (y+1)^2$ shows it is symmetrical around $x = 0$ and $y = -1$. So: (b).

4. MAT 2014 b

   Symmetry: rewriting the equation as $y = \log_{10}((x-1)^2 + 1)$ shows it is symmetrical around $x = 1$. **Simple special case**: when $x = 1$, $y = 0$. So: (e).

5. MAT 2012 j

   Simple special case: when $\theta = \frac{\pi}{2}$, PQR is in a semicircle, and QR is a diameter, so area = $1 + \frac{\pi}{2}$. So: (b).

6. MAT 2014 d

   Simple special case: consider when m is very large. Then the line $y = mx$ is almost the y-axis, and the reflection is almost (-1,0). The only formula which fits is (d).

7. MAT 2014 f

   TT brings us back where we started (so it's a **symmetry** transformation), and so does STST or TSTS. Either way, that's adding an even number of Ts and an even number of Ss. We must have an odd number of Ts in order to get the minus in $8 - x$, and eight Ss (plus an even number) to get the 8. (c): odd number of Ts, even number of Ss.

8. MAT 2010 b

   Simple special case: when n=0, the sum is 0, which rules out all except (a) and (d). When n=1, the sum is 2, which means it must be (a).

9. MAT 2014 h

   Symmetry: F goes through the same pattern of increase in every cycle of six numbers, n=1 to 6, n=7 to 12, n=13 to 18, etc. The increase over each cycle of six numbers is the same. So the answer must be a multiple of 1000; so, (c).

10. MAT 2014 j

    $\int_{-1}^{1} f(t)\,\mathrm{d}t$ is just a number. Call it A. The interval $[-1,1]$ is **symmetrical** around x=0, so the integral of f(−x) from −1 to 1 is the same as the integral of f(x) from −1 to 1. It is A. Integrate the given equation between the bounds -1 and 1.

    $12 + A = 2A + 3A\int_{-1}^{1} x^2\,\mathrm{d}x$, so A=4

11. MAT 2013 h

    The area is a sector of a circle minus a triangle, so a circle-type area minus a triangle-type area, so a "π" bit minus something with "$\sqrt{2}$" bits. So it's (b).

12. MAT 2013 i

    F repeats its values, so all its values are either 1 or −1. The −1 values come in patches of $1, 2, 4, 8\ldots$ (each patch doubling the previous one). Up to 100, you get the patches 1, 2, 4, 8, 16 (total 31), plus the first five of the 32-patch, in all 36 −1 values. $64 \times 1 + 36 \times -1$ gives us 28, or (c).

13. STEP 1999 Q.1

    First considering a simplified case: Consider integers greater than or equal to zero and less than ten. $9, \cancel{8}, 7, \cancel{6}, \cancel{5}, \cancel{4}, 3, \cancel{2}, 1, \cancel{0}$

    Four of those 10 integers not divisible by 2 or by 5. But the same will be true of every group of 10. So the number of the integers greater than or equal to 0, and less than a million, which are not divisible by 2 or by 5, is 400,000.

    If you add $\cancel{10}$, this list is **symmetrical**, but also unchanged, so its average is 5. Then every group of 10 you add leaves the list, with the top number added and then cancelled ($\cancel{20}, \cancel{30}, \ldots$) still symmetrical. So the average for the list up to one million is 500,000.

    $4179 = 21 \times 199$, so this is a similar problem to the previous one, only we should consider sub-groups of 21.

    $20, 19, \cancel{18}, 17, 16, \cancel{15}, \cancel{14}, 13, \cancel{12}, 11, 10, \cancel{9}, 8, \cancel{7}, \cancel{6}, 5, 4, \cancel{3}, 2, 1, \cancel{0}$

    12 of these integers not divisible, so the number of not-divisible integers in the list up to 4179 is $12 \times 199 = 2388$. Again, **symmetrical**, so average for those up to 21 is $\frac{21}{2}$, and average for the list up to 4179 is $2089\frac{1}{2}$.