Session 1

1. Into a maximum of how many regions do n lines divide a plane?

Learning methods

Drawing a diagram and trying it out with one lines, two lines, three lines is the best start.

Possible follow-up: why the *maximum* number? When can you have fewer? What is the *minimum* number?

You can do this by observing that when you draw the n'th line, it crosses a maximum of (n-1) lines and creates n new regions. There is one region even if you draw zero lines, so maximum with n lines is $1+1+2+3+\ldots+n$. Neater is as follows:

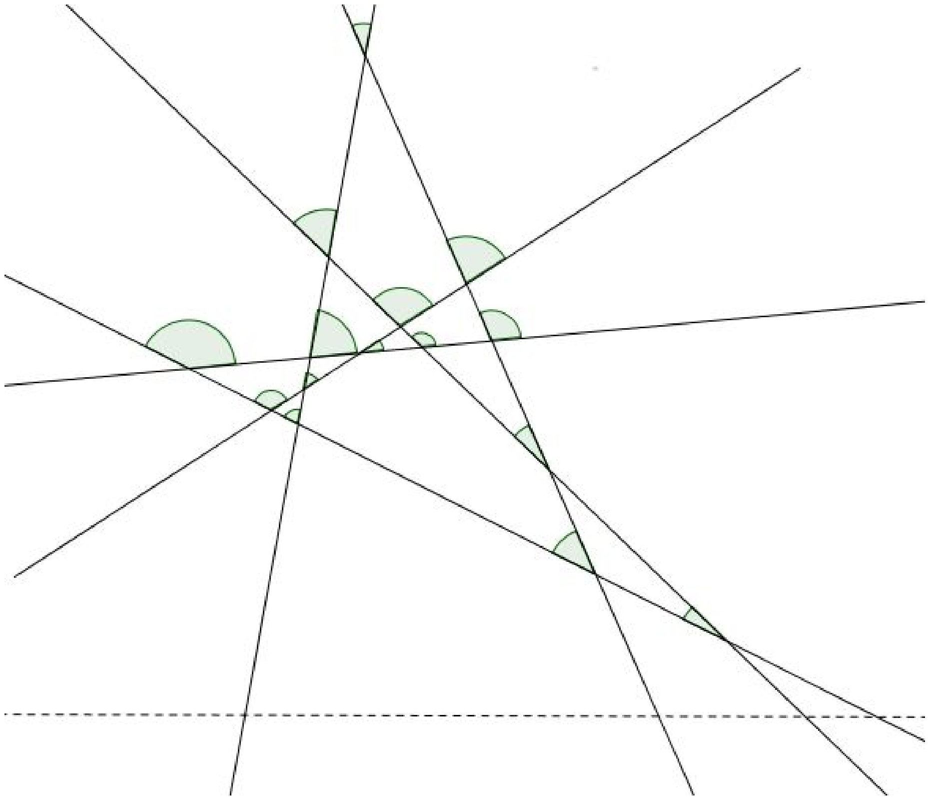
If necessary, tilt the plane microscopically so that every meeting-point of lines is at a different height. Draw a horizontal line across the plane below all the meeting-points of lines.

Above the horizontal line, the regions which have a lowest point all have a unique lower vertex, and each vertex is the lowest point of a region.

So number of regions above the line = number of vertices = number of meeting-points of lines = nC2

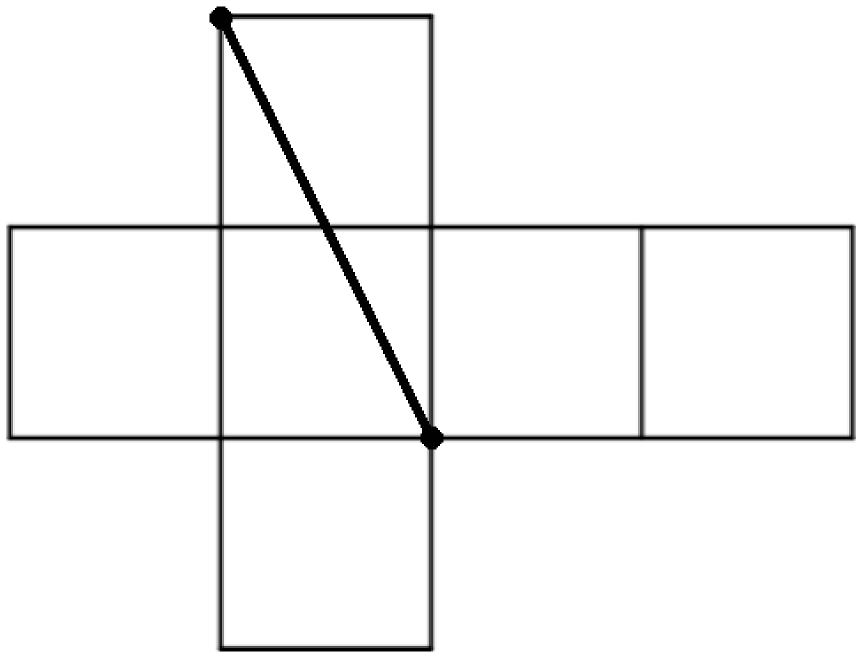
Below the horizontal line, there are n+1 regions which have no lowest point.

Total number of regions = nC2 + n + 1 = sum of first three numbers in n'th row of Pascal's triangle.

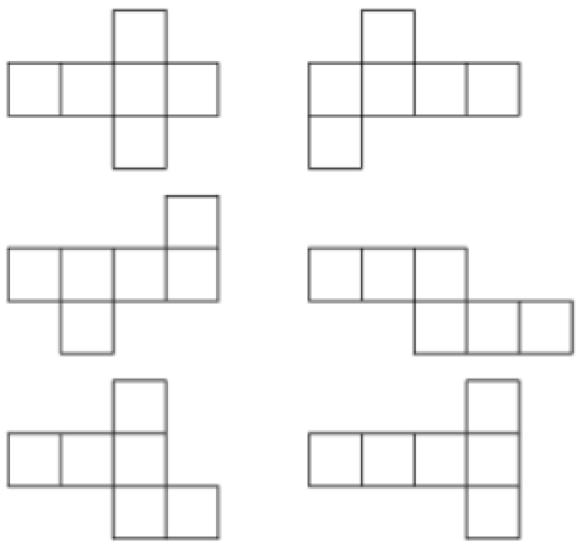


2. What is the length of the shortest route for an ant from a vertex of a cube to the opposite vertex?

Again, a diagram is the key. Draw the net of the cube.



Possible follow-up: how many distinct nets for the cube (not counting reflections as distinct)? (Answer: 11)



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3. Ashley throws a ball aiming directly at Brenda, who is perched in a tree. At exactly the same time, Brenda falls from the tree. Can she catch the ball?

Assume this happens on a planet with negligible gravity. Then the ball will travel on a straight line as aimed, and Brenda will stay where she is, so she certainly catches it.

Now consider how the picture is changed as g increases above zero. At the time t when the ball reaches Brenda's line of fall, the ball will be $\frac{1}{2}gt^2$ below where it would have been on the straight-line trajectory, and Brenda will be exactly the same distance below her perch. So, yes, she can catch it.

You might be asked your assumptions. You're assuming that air resistance is negligible; or at least of similar effect for the ball and for Brenda; or at least that the difference in the effects is small compared to Brenda's ability to catch balls slightly higher or lower.

Also, that the ball is thrown fast enough that the ball and Brenda do not hit the ground before the ball can reach Brenda's line of fall.

And that there isn't a high wind which blows Brenda out of the plane of the no-wind version of the problem faster than it blows the ball out of that plane.

Session 2

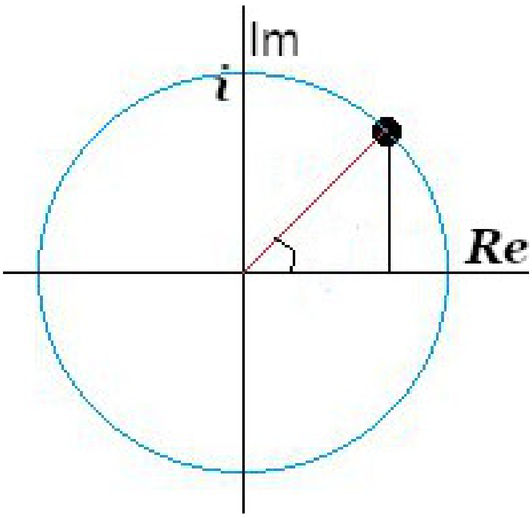
1. What are the square roots of i?

$i = \mathrm{e}^{\frac{i\pi}{2}}$, so

$\sqrt{i} = \pm\mathrm{e}^{\frac{i\pi}{4}} = \pm\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$

Learning methods

This answer is most easily seen from a diagram.



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2. Prove $n^3 - n$ is divisible by 6 for all n

This can be done by induction, but quicker by factorising.

$n^3 - n = n(n-1)(n+1)$

$(n-1), n, (n+1)$ are three successive whole numbers, so at least one of them is divisible by 2 and at least one is divisible by 3. So their product is divisible by $2 \times 3 = 6$

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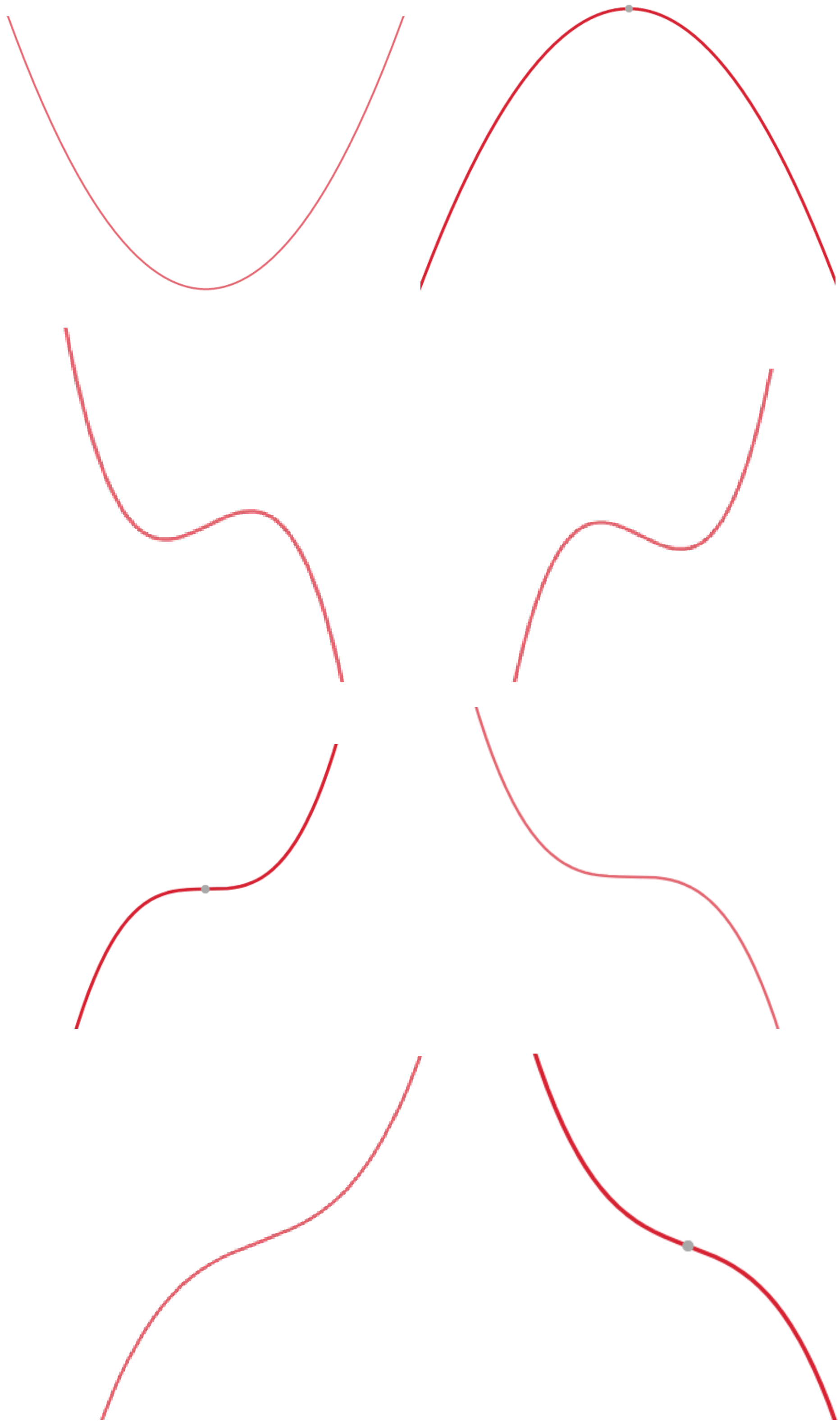
3. Sketch the basic shapes for the quadratic curve $y = ax^2 + bx + c$ $(a \neq 0)$. Sketch the basic shapes for the cubic curve $y = ax^3 + bx^2 + cx + d$ $(a \neq 0)$.



Learning methods

You can often get a good sketch from asymptotic behaviour and turning points. This question also tests systematically considering cases.

Session 3

1. $c^2 = a^2 + b^2$ and a, b, c are all whole numbers. Prove that they can't all be odd (in other words, at least one must be even). Then prove that at least one of a and b must be even. Then prove that at least one of a and b must be divisible by 4.

You can do this best by using proof by contradiction.

$\text{Odd}^2 + \text{Odd}^2 = \text{Even}$, so it's impossible for a, b, c all to be odd.

If a and b are both odd, then $a^2 = (2n+1)^2 = 4n^2+4n+1$ for some n, and $b^2 = (2m+1)^2 = 4m^2+4m+1$ for some m, so $c^2 = a^2 + b^2 = 4(n^2 + m^2 + n + m) + 2$ and is an even number *not* divisible by 4.

c cannot be odd (because then c^2 would be odd). Nor can it be even (because then c^2 would be an even number divisible by 4).

So, by contradiction, we have proved that a and b cannot be both odd.

Possible follow-up: whichever of a and b is even must be not just even, but also divisible by 4. (Consider 3,4,5; 5,12,13; 7,24,25; 8,15,17, etc.)

Proof by contradiction.

Without loss of generality, we can consider triples a, b, c in lowest terms. Then suppose a is the even one. If a is not divisible by 4, then $a = 4n + 2$ for some n, and $b = 4m + 1$ or $b = 4m + 3$ for some m.

$a^2 + b^2 =$

(something divisible by 8) +4+ (something divisible by 8) +1 = (something divisible by 8) +5

But whatever c is, its square can't leave remainder 5 when divided by 8.

So we have a contradiction, and in fact $4|a$

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2. If I had a cube and six colours and painted each face a different colour, how many (different) ways could I paint the cube? What about if I had n colours instead of 6?

First up, since the problem says nothing about colours on faces next to each other, or opposite each, it doesn't matter that we're dealing with a cube specifically. We're just dealing with six areas.

Number them. 6 choices for area 1, 5 for area 2, and so on. A total of $6 \times 5 \times 4 \times 3 \times 2 \times 1$, or 6!, choices.

With n colours, $n \times (n-1) \times (n-2) \times (n-3) \times (n-4) \times (n-5)$

This is assuming that the cube is in a fixed position, so a colouring with red at the bottom counts as different from the same colouring turned upside down with red at the top. If you can rotate the cube, then you must divide each answer by 24.

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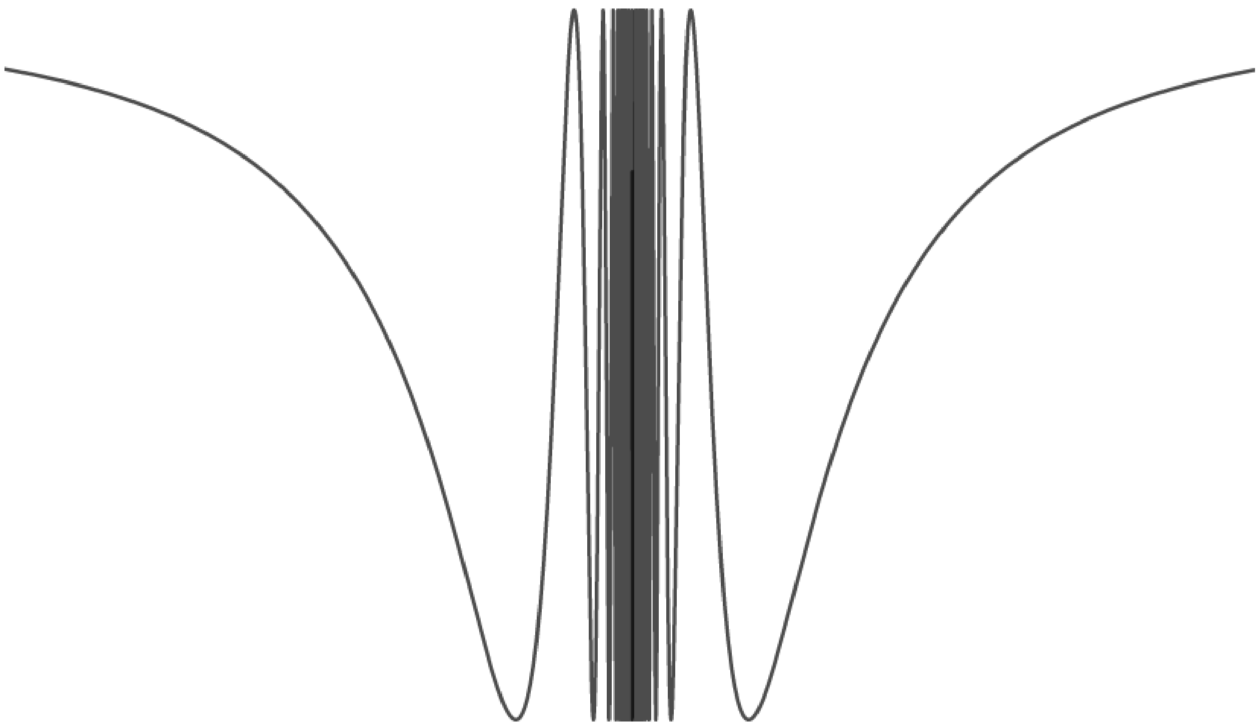
3. Sketch the graph of $y = \cos\frac{1}{x}$.



You know that y only varies between -1 and 1

Since $\cos x$ approaches 1 as x approaches 0, you know that y just gets closer and closer to 1 for big positive or big negative values of x

Since $\cos t = \cos -t$ for all t, you know the graph is symmetrical around $x = 0$

Since $\cos t$ varies between -1 and 1 as t varies between $2n\pi$ and $(2n+1)\pi$, (for all n), you know y will oscillate between -1 and 1 faster and faster for small values of x

Possible follow-up: what are those last two points on each side of $x = 0$ where $y = -1$, and beyond which the curve stops oscillating?

The turning-point for positive x after which y stops oscillating and just increases is given by $x = \frac{1}{t}$ if $x = t$ is the turning point *before* which $\cos x$ does not oscillate but only increases. That turning point for $\cos x$ is at $x = \pi$, so the corresponding turning point for $\cos\frac{1}{x}$ is at $x = \frac{1}{\pi}$
