2 Diagrams, special cases, and systematically considering cases

Starters

- *Prove* that the angle subtended by an arc at the centre of a circle is twice the angle subtended at the circumference.
- Find all sets of positive integers a, b that satisfy the equation

$$\frac{1}{a}+\frac{1}{b}=1$$

- Siklos APM 2008 edition p.1 Q.1 - (i) Find all sets of positive integers a, b, c that satisfy the equation

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=1$$

Learning objectives

- Developing a habit of doing big, clear diagrams. (Sometimes we draw small, obscure diagrams with the semi-conscious motive that if the diagram is small and obscure, then our mistakes will be invisible. But that's exactly why we should do big, clear diagrams: to be able to see our own mistakes and correct them).
- Thinking of *different* diagrams for the *same* problem, so we can find the diagram that shows the solution or makes it easiest.
- Getting a start on problems, or checking your work, by considering *simple special cases*.
- Dealing with problems by breaking them down systematically into different cases (as in proof by exhaustion)

Introduction

PowerPoint at bit.ly/diag-ppt

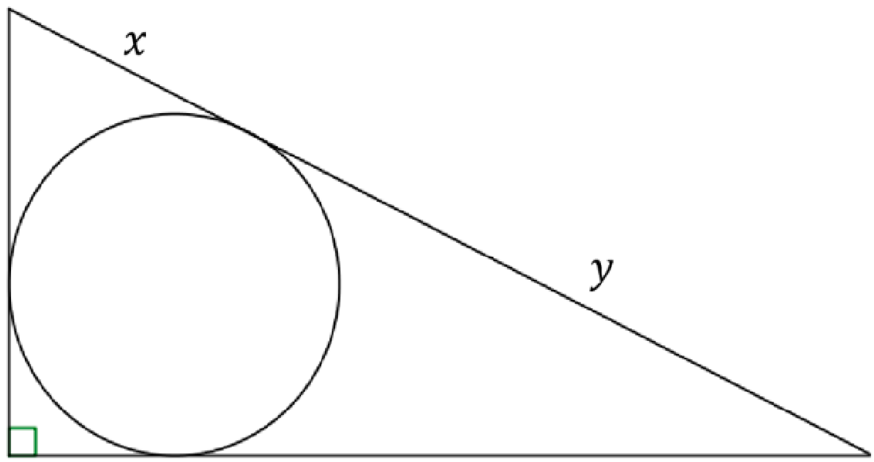
Practice

1. **I/2001/1** The points A, B, and C lie on the sides of a square of side 1 cm and no two points lie on the same side. Show that the length of at least one side of the triangle ABC must be less than or equal to $\sqrt{6}-\sqrt{2}$ cm.

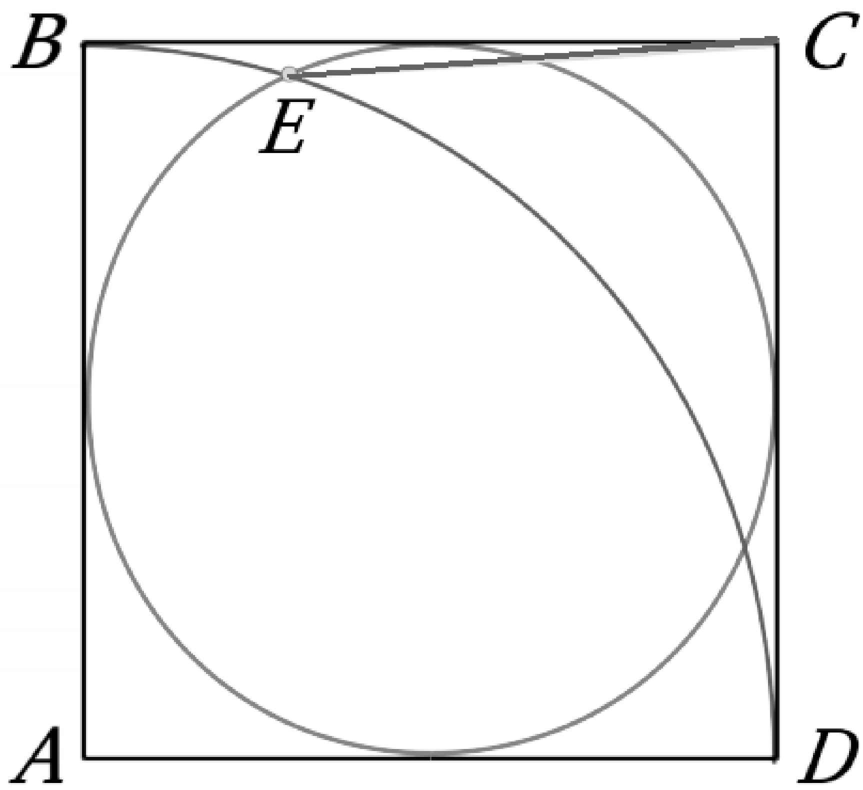
   bit.ly/1-2001-1

2. **Assignment 21 from the online STEP preparation course**

   (i) Find the area of this triangle in terms of x and y



   (ii) Prove EC is half the length of the diagonal of the square



   bit.ly/assig21

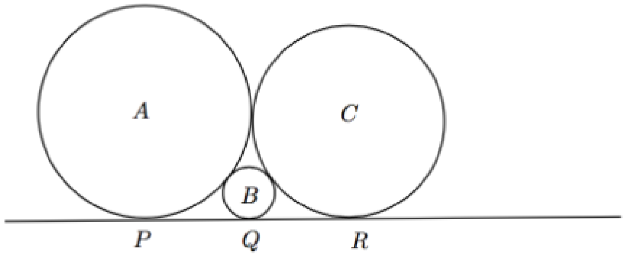
3. **I/2006/2** A small goat is tethered by a rope to a point at ground level on a side of a square barn which stands in a large horizontal field of grass. The sides of the barn are of length $2a$ and the rope is of length $4a$. Let A be the area of the grass that the goat can graze. Prove that $A \leq 14\pi a^2$ and determine the minimum value of A.

bit.ly/1-2006-2

I/2016/5.

4. bit.ly/1-2016-5

(i)



The diagram shows three touching circles A, B and C, with a common tangent PQR. The radii of the circles are a, b and c, respectively.

Show that

$$\frac{1}{\sqrt{b}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{c}} \qquad (*)$$

and deduce that

$$2\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) = \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^2 . \qquad (**)$$

(ii) Instead, let a, b and c be positive numbers, with $b < c < a$, which satisfy $(**)$. Show that they also satisfy $(*)$.

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5. **I/2015/3** A prison consists of a square courtyard of side b bounded by a perimeter wall and a square building of side a placed centrally within the courtyard. The sides of the building are parallel to the perimeter walls. Guards can stand either at the middle of a perimeter wall or in a corner of the courtyard. If the guards wish to see as great a length of the perimeter wall as possible, determine which of these positions is preferable. You should consider separately the cases $b < 3a$ and $b > 3a$

   bit.ly/step-1stspiral.

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6. **I/1999/13**

   Bar magnets are placed randomly end-to-end in a straight line. If adjacent magnets have ends of opposite polarities facing each other, they join together to form a single unit. If they have ends of the same polarity facing each other, they stand apart. Find the expectation and variance of the number of separate units in terms of the total number N of magnets.

   (For this one you need to know a little, but only a little, about the binomial distribution. The "expectation" or "expected value" is the mean value you would get from a very large number of experiments. The "variance" is the mean squared deviation from the expected value. ["Standard deviation" is square root of variance]. bit.ly/prob-mech.

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7. **I/2007/5** - *Note: a regular octahedron is a polyhedron with eight faces each of which is an equilateral triangle.*

   (i) Show that the angle between any two faces of a regular octahedron is $\arccos\left(-\frac{1}{3}\right)$

   (ii) Find the ratio of the volume of a regular octahedron to the volume of a cube whose vertices are the centres of the faces of the octahedron.

   bit.ly/1-2007-5

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8. **II/1999/14**

   You play the following game. You throw a six-sided fair die repeatedly. You may choose to stop after any throw, except that you must stop if you throw a 1. Your score is the number obtained on your last throw. Determine the strategy that you should adopt in order to maximize your expected score, explaining your reasoning carefully. (bit.ly/prob-mech).

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9. I/2007/1 - A positive integer with $2n$ digits (the first of which must not be 0) is called a balanced number if the sum of the first n digits equals the sum of the last n digits. For example, 1634 is a 4-digit balanced number, but 123401 is not a balanced number.

   (i) Show that seventy 4-digit balanced numbers can be made using the digits 0, 1, 2, 3 and 4.

   (ii) Show that $\frac{1}{6}k(k+1)(4k+5)$ 4-digit balanced numbers can be made using the digits 0 to k. You may use the identity $\sum_1^n r^2 \equiv \frac{1}{6}n(n+1)(2n+1)$

   bit.ly/1-2007-1x

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10. I/2009/1 - A proper factor of an integer N is a positive integer, not 1 or N, that divides N. (i) Show that $3^2 \times 5^3$ has exactly 10 proper factors. Determine how many other integers of the form $3^m \times 5^n$ (where m and n are integers) have exactly 10 proper factors. (ii) Let N be the smallest positive integer that has exactly 426 proper factors. Determine N, giving your answer in terms of its prime factors.

    bit.ly/1-2009-1
