III/2008/8

$P$ is $\lambda \underline{a} + (1-\lambda)\underline{b}$

If $A\hat{O}P = B\hat{O}P$, then $\cos A\hat{O}P = \cos B\hat{O}P$, so

$$\frac{\vec{OA}\cdot\vec{OP}}{ap} = \frac{\vec{OB}\cdot\vec{OP}}{bp} \quad \text{if } p = |\vec{OP}|$$

$$b\,\underline{a}\cdot(\lambda\underline{a} + (1-\lambda)\underline{b}) = a\underline{b}\cdot(\lambda\underline{a} + (1-\lambda)\underline{b})$$

$$\lambda(ba^2 - (a+b)\,\underline{a}\cdot\underline{b} + ab^2) = ab^2 - b\,\underline{a}\cdot\underline{b}$$

$$\lambda\cdot(a+b)(ab - \underline{a}\cdot\underline{b}) = b(ab - \underline{a}\cdot\underline{b})$$

unless $A=B$, $\lambda = b/(a+b)$

$Q$ is $(1-\lambda)\underline{a} + \lambda\underline{b}$ by symmetry

$$\text{so } OQ^2 = \vec{OQ}\cdot\vec{OQ} = (1-\lambda)^2a^2 + \lambda^2b^2 + 2\lambda(1-\lambda)\,\underline{a}\cdot\underline{b}$$

$$OP^2 = \lambda^2a^2 + (1-\lambda)^2b^2 + 2\lambda(1-\lambda)\,\underline{a}\cdot\underline{b}$$

$$OQ^2 - OP^2 = (1-2\lambda)(a^2-b^2)$$

$$= \frac{a-b}{a+b}(a^2-b^2) = (a-b)^2. \quad \square$$