II/1996/3 - The Fibonacci numbers F_n are defined by the conditions $F_0 = 0$, $F_1 = 1$ and

$$F_{n+1} = F_n + F_{n-1}$$

for all $n \geqslant 1$. Show that $F_2 = 1$, $F_3 = 2$, $F_4 = 3$ and compute F_5, F_6 and F_7.

Compute $F_{n+1}F_{n-1} - F_n^2$ for a few values of n; guess a general formula and prove it by induction, or otherwise.

By induction on k, or otherwise, show that

$$F_{n+k} = F_k F_{n+1} + F_{k-1} F_n$$

for all positive integers n and k.

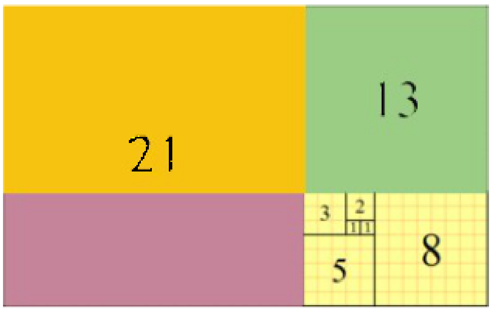
$F_2 = F_1 + F_0 = 1;\ F_3 = F_2 + F_1 = 2;\ F_4 = F_3 + F_2 = 3;\ F_5 = F_4 + F_3 = 5;\ F_6 = F_5 + F_4 = 8;\ F_7 = F_6 + F_5 = 13$

$F_2 \cdot F_0 - (F_1)^2 = 0 - 1 = -1$

$F_3 \cdot F_1 - (F_2)^2 = 2 - 1 = 1$

$F_4 \cdot F_2 - (F_3)^2 = 3 - 4 = -1$

Guess: $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$



Proof: $F_{n+1}F_{n-1} - F_n^2$ = rectangle made up of green plus yellow area minus square made up of pink plus yellow areas = green area minus pink area $= -(F_n F_{n-2} - F_{n-1}^2)$. Since $F_2 \cdot F_0 - F_1^2 = 0 - 1 = -1$, the conclusion follows.

To prove $F_{n+k} = F_k F_{n+1} + F_{k-1} F_n$ by induction

Basis step: prove true for $k = 1$ and $k = 2$

$F_1 F_{n+1} + F_0 F_n = 1 \cdot F_{n+1} + 0 \cdot F_n = F_{n+1}$ $\square$

$F_2 F_{n+1} + F_1 F_n = F_{n+1} + F_n = F_{n+2}$ $\square$

Induction step: prove true for $k = r + 1$ if true for $k = r$ and $k = r - 1$

$F_{n+r+1} = F_{n+r} + F_{n+r-1}$ by definition

$\ldots = F_r F_{n+1} + F_{r-1} F_n + F_{r-1} F_{n+1} + F_{r-2} F_n$ if claim true for $k = r$ and $k = r - 1$

$\ldots = F_{r+1} F_{n+1} + F_r F_n$ $\square$