Number theory / Analysis

S3 2015 Q5

Assume $\sqrt{2}$ is rational, that is $\sqrt{2} = \frac{p}{q}$ where $p, q \in \mathbb{Z}$ and $\gcd(p,q) = 1$

Let $S = \{ n \in \mathbb{Z}^+ \mid n\sqrt{2} \in \mathbb{Z}^+ \}$

Since $q \cdot \sqrt{2} = q \cdot \frac{p}{q} = p \in \mathbb{Z}^+$, $q \in S$ so it is non-empty

Let $k \in S$ and $k \leq n \ \forall n \in S$.

$(\sqrt{2} - 1)k = \sqrt{2}k - k$

since $k \in \mathbb{Z}^+$ and $\sqrt{2}k \in \mathbb{Z}^+$ $\sqrt{2}k - k \in \mathbb{Z}$

So, $\sqrt{2}k - k \in S$ since $k\sqrt{2} < 2k$

Since $k\sqrt{2} - k \in S$ and $k\sqrt{2} - k < k$ ✓ it contradicts the fact that k was the smallest integer in S.

Hence, $\sqrt{2}$ is irrational

(ii) Assume $2^{\frac{1}{3}}$ is rational then

$2^{\frac{1}{3}} = \frac{p}{q}$ where $\gcd(p,q) = 1$

$2^{\frac{2}{3}} = \frac{p^2}{q^2} \in \mathbb{Q}$. Hence $2^{\frac{1}{3}} \in \mathbb{Q}$ implies $2^{\frac{2}{3}} \in \mathbb{Q}$

Conversely, assume $2^{\frac{2}{3}} \in \mathbb{Q}$ then

$2^{\frac{2}{3}} = \frac{p}{q}$ where $\gcd(p,q) = 1$

$(2^{\frac{1}{3}})^2 = 2^{\frac{2}{3}} = \frac{p}{q}$

$2^{\frac{4}{3}} = \frac{p^2}{q^2} \Rightarrow 2 \cdot 2^{\frac{1}{3}} = \frac{p^2}{q^2}$

$2^{\frac{1}{3}} = \frac{p^2}{2q^2} \in \mathbb{Q}$

Hence, $2^{\frac{2}{3}} \in \mathbb{Q} \Rightarrow 2^{\frac{1}{3}} \in \mathbb{Q}$.

Suppose $2^{\frac{1}{3}}$ is rational, that is $2^{\frac{1}{3}} = \frac{p}{q}$, $\gcd(p,q) = 1$

Let $S = \{ n \in \mathbb{Z}^+ \text{ st } n\sqrt[3]{2} \in \mathbb{Z}^+ \}$

Since $q \cdot \sqrt[3]{2} = p \in \mathbb{Z}^+$ $q \in S$

Let $k \in S$ such that $k \leq n$ $\forall n \in S$.

$(\sqrt[3]{2} - 1)k = k\sqrt[3]{2} - k \in S$ and $k\sqrt[3]{2} - k < k$ since $k\sqrt[3]{2} < 2k$

Contradicting the fact that k was the smallest integer in S. Hence, $2^{\frac{1}{3}}$ is irrational $\Rightarrow 2^{\frac{2}{3}}$ is irrational.