# Advanced Problems in Mathematics

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**43 problems complete with full solutions and discussion**

# CONTENTS



Each problem has been given a title (not used later on), to remind you what it is about if you have already looked at it, followed by a rough indication of the mathematical content in case you want to pick out questions by topic.

# PREFACE

I have set out below the answers to the questions you will want to ask before getting down to work. If I have forgotten anything, let me know and I will include it in the next of what I hope will be many editions.

**Is this booklet suitable for me?** The booklet is intended for anyone taking A-level mathematics or the equivalent (Baccalaureat or CSYS, for example) who wants to try something more challenging than A-level style questions. However, many questions involve topics which occur in Further Mathematics syllabuses. Furthermore, even if you are taking Further Mathematics there are sure to be some questions on topics which are not in your particular syllabus since syllabuses differ widely between examination boards. You will therefore have to be selective in your choice of questions.

**What is the purpose of this booklet?** There is considerable feeling amongst university teachers that the present A-levels, though excellent in many ways, fail to equip students with some of the techniques required for a university mathematics course, or indeed for courses such as engineering, physical sciences and economics which depend on mathematics. The first purpose of this booklet is to offer an opportunity for such students to look into a new world of mathematics. The second purpose is to give support to students who are planning to take STEP. It will be especially useful for students in schools which cannot, for lack of time or staff, supply high level teaching in mathematics.

**What is STEP?** STEP is an examination administered by Cambridge Assessment. It is used as a basis for conditional offers in Mathematics by Cambridge and Warwick universities, but the papers are taken by many students who do not hold offers from these universities'. Anyone *can* enter and it is my view that good mathematicians *should* enter. STEP should be regarded as a challenge in the same spirit as the national mathematics competitions and olympiads which are so popular. However, STEP provides a very different sort of challenge from these competitions, since it concentrates on in-depth problems in mainstream (e.g. A-level) mathematics.

**Where did the questions come from?** Mostly from old STEP papers. In fact, the great majority are questions I set myself when I was a STEP examiner. Others were set by my colleagues Tom Körner, Michael Potter, Dennis Barden and the Rae Mitchell. However, some of the questions have a long history (maybe all of them: it is sometimes said that there is no such thing as a new mathematics question), so I do not think any of us would claim any special credit for our efforts.

I chose mostly my own questions not because I think that they are the best and not because I can do them. (At least, not just for these reasons.) I chose them because I have retained a clear idea of what I was looking for when I set them and this makes the discussion and solution much easier to provide.

**Will I find the questions difficult?** I hope so. If I have provided a set of 43 questions that you can romp through, then I have wasted my time. Do not be discouraged if you are struggling with a question; the most worthwhile things are learnt the hard way. It is important that you adjust your sights. A typical A-level question may take, say, ten minutes and consist of a single guided step ('Use the substitution $x = \cdots$ to evaluate the following integral'). The questions in this booklet are multi-step, and often you have to work through many steps without guidance. If you do four STEP questions reasonably well in the three-hour examination you are doing well. You must keep this comparison firmly in your mind.

**Are all the questions of a similar level of difficulty?** No. Not surprisingly, many of the original STEP questions were more suitable for examinations than for private study, and I did not hesitate to adapt a question where I saw the need or where I saw an opportunity to make it more interesting. Sometimes, this made the question significantly longer. As a result of my meddling, the questions are therefore more uneven in standard than could be tolerated on an examination paper.

**Are the questions in order of difficulty?** No. I set out to order them but found that everyone had different ideas about which questions were easy. I suppose it just depends on individual experiences. The last three questions are not tests for intellectual giants: they are the ones which would not fit on two sides of paper. However, I do think that the first ten or so will be more amenable to students who are not taking A-level Further Mathematics (or the equivalent), while some of the later questions are decidedly tricky.

**Why is there a discussion before each solution?** Two reasons. First, I tried to anticipate difficulties which might prevent you starting the question and used the discussion section to supply appropriate mild hints. Second, I have provided some background information, either historical or mathematical, to give you some insight into the origin, purpose and/or mathematical significance of the question. The level of the discussions vary; in some cases, quite sophisticated mathematical ideas are introduced which will appeal only to the most dedicated readers.

**Are the solutions all perfect?** Yes. Well, maybe not. They were checked at an early stage by first year undergraduates and at a much later stage by teachers who attended the annual Cambridge Colloquium in 1995. (Thank you very much for your considerable efforts, if you were one of those who helped with the proofreading.) However, I have since made a number of improvements in response to suggestions, so it is not beyond the bounds of possibility that I have introduced new errors. (In fact, it is extremely likely, judging by past form.) I hope that there is nothing so serious that you are significantly inconvenienced.

**What should I do if I find a mistake?** E-mail me at stcs@cam.ac.uk. You do not have to find a mistake in order to contact me: let me know also if you have a better way of solving a problem or if you just have something interesting to tell me about a question.

**Can I use a calculator?** One of the errors in my solutions that I had to correct involved the numbers in question 7; someone kindly referred to it as 'just a calculator error', though it was actually a muddle with a previous version. That prompts me to tell you that *calculators are not required* for any of these questions; my advice is to remove the battery so that you are not tempted. Note that *calculators are not permitted* in STEP.

**And finally** ... I have enjoyed producing this booklet, despite the frustrations of many revisions due to explanations that I thought were crystal clear but which turned out on rereading to be muddily opaque. I very much hope that you will find the questions not only useful but also interesting, entertaining and even (sometimes) amusing.

Stephen Siklos, October 2008

**Question 1**

(i) Find all sets of positive integers $a$, $b$ and $c$ that satisfy the equation

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1.$$

(ii) Determine the sets of positive integers $a$, $b$ and $c$ that satisfy the inequality

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geqslant 1.$$

**Discussion**

Age has not diminished the value of this old chestnut. It requires almost no mathematics in the sense of GCSE or A-level syllabuses, but instead tests a vital asset for a mathematician, namely the capacity for systematic thought. For this reason, it tends to crop up in mathematics contests, where competitors come from different backgrounds. I most recently saw it in the form 'Find all the positive integer solutions of $bc + ca + ab = abc$'. (You see the connection?)

You will want to make use of the symmetry between $a$, $b$ and $c$: if, for example, $a = 2$, $b = 3$ and $c = 4$, then the same solution can be expressed in five other ways, such as $a = 3$, $b = 4$ and $c = 2$. You do not want to derive all these separately, so you need to order $a$, $b$ and $c$ in some way.

The reason the question uses the word 'determine', rather than 'find' for part (ii) is that in part (i) you can write down all the possibilities explicitly, whereas for part (ii) there are, in some cases, an infinite number of possibilities which obviously cannot be explicitly listed, though they can be described and hence determined.

**Solution to Question 1**

Let us take, without any loss of generality, $a \leqslant b \leqslant c$.

(i) How small can $a$ be?

First set $a = 1$. This gives no solutions, because it leaves nothing for $1/b + 1/c$.

Next set $a = 2$ and try values of $b$ (with $b \geqslant 2$, since we have assumed that $a \leqslant b \leqslant c$) in order:
if $b = 2$, then $1/c = 0$, which is no good;
if $b = 3$, then $c = 6$, which works;
if $b = 4$, then $c = 4$, which works;
if $b \geqslant 5$, then $c \leqslant b$, so we need not consider this.

Then set $a = 3$, and try values of $b$ (with $b \geqslant 3$) in order:
if $b = 3$, then $c = 3$, which works;
if $b \geqslant 4$, then $c \leqslant b$, so we need not consider this.

Finally, if $a \geqslant 4$, then at least one of $b$ and $c$ must be $\leqslant a$, so we need look no further.

The only possibilities are therefore $(2, 3, 6)$, $(2, 4, 4)$ and $(3, 3, 3)$.

(ii) Clearly, we must include all the solutions found in part (i). We proceed systematically, as in part (i).

First set $a = 1$. This time, any values of $b$ and $c$ will do.

Next set $a = 2$ and try values of $b$ (with $b \geqslant 2$, since we have assumed that $a \leqslant b \leqslant c$) in order:
if $b = 2$, then any value of $c$ will do.
if $b = 3$, then 3, 4, 5 and 6 will do for $c$, but 7 is too big;
if $b = 4$, then $c = 4$ will do, but 5 is too big for $c$;
if $b \geqslant 5$, then $c \leqslant b$, so we need not consider this.

Then set $a = 3$ and try values of $b$ (with $b \geqslant 3$) in order:
if $b = 3$, then $c = 3$, which works, but $c = 4$ is too big;
if $b \geqslant 4$, then $c \leqslant b$, so we need not consider this.

As before $a = 4$ and $b \geqslant 4$, $c \geqslant 4$ gives no possibilities.

Therefore the extra sets for part (ii) are of the form $(1, b, c)$, $(2, 2, c)$, $(2, 3, 3)$, $(2, 3, 4)$ and $(2, 3, 5)$.

A slightly different approach for part (i), which would also generalise for part (ii), is to start with the case $a = b = c$, for which the only solution is $a = b = c = 1/3$. If $a$, $b$ and $c$ are not all equal then one of them, which we may take to be $a$, must be greater than $1/3$, i.e. $1/2$. The two remaining possibilities follow easily.

**Question 2**

(i) Write down the average of the integers $n_1$, $(n_1+1)$, $\dots$ , $(n_2-1)$, $n_2$. Show that

$$n_1 + (n_1+1) + \cdots + (n_2-1) + n_2 = \tfrac{1}{2}(n_2 - n_1 + 1)(n_1 + n_2).$$

(ii) Write down and prove a general law of which the following are special cases:

$$\begin{aligned} 1 &= 0+1 \\ 2+3+4 &= 1+8 \\ 5+6+7+8+9 &= 8+27 \\ 10+11+\cdots+16 &= 27+64. \end{aligned}$$

Hence prove that

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \tfrac{1}{4}n^2(n+1)^2.$$

**Discussion**

For comments on 'write down a general rule', see Question 4.

You could use induction to prove your general law but it is not necessary. The obvious alternative involves use of part (i). You will also need part (i) for the second bit of part (ii). You might think of induction for the second bit of part (ii), but the question specifically tells you to use the previous result.

In fact, there are various ways of summing $k$th powers of integers (besides the trick given here, which does not readily generalise to powers other than the third). A simple way is to assume that the result is a polynomial of degree $k+1$. You can then find the coefficients by substituting in the first $k+1$ cases. For example, for cubes, let $\sum_1^n r^3 = a_4n^4 + a_3n^3 + a_2n^2 + a_1n + a_0$. Setting $n=1$ gives $1 = a_4 + a_3 + a_2 + a_1 + a_0$. Similarly, setting $n = 2, 3, 4, 5$ gives 5 linear equations for the five unknowns $(a_4, \dots , a_0)$. We can guess that the leading coefficient will in general be $(k+1)^{-1}$, because the sum approximates the area under the graph $y = x^k$ from $x = 1$ to $x = n$, which is given by the integral of $x^k$.

Another way is to use the difference method, which is based on the identity

$$\sum_{r=1}^{n} \mathrm{f}(r+1) - \mathrm{f}(r) = \mathrm{f}(n+1) - \mathrm{f}(1).$$

You just need a cunning choice of $\mathrm{f}(r)$. Try it with

$$\mathrm{f}(r) = \tfrac{1}{2}r(r-1) \quad \text{and} \quad \mathrm{f}(r) = \tfrac{1}{3}(r+1)r(r-1).$$

You should then be able to work out a generalisation which will give you $\sum_1^n r^3$.

**Solution to Question 2**

(i) The average is $\frac{1}{2}(n_1 + n_2)$. (This is the mid-point of a ruler with $n_1$ at one end and $n_2$ at the other.) The sum of any numbers is the average of the numbers times the number of numbers, as given.

(ii) A general rule is

$$\sum_{k=m^2+1}^{(m+1)^2} k \ = m^3 + (m+1)^3.$$

We can prove this result by applying the formula derived in the first part of the question:

$$\begin{aligned}
\sum_{n=m^2+1}^{(m+1)^2} k &= \tfrac{1}{2}\big[(m+1)^2 - m^2\big]\big[(m+1)^2 + (m^2+1)\big] &(1)\\
&= \tfrac{1}{2}\big[2m+1\big]\big[2m^2+2m+2\big] &(2)\\
&= 2m^3 + 3m^2 + 3m + 1 &(3)\\
&= m^3 + (m+1)^3. &(4)
\end{aligned}$$

For the last part, we can obtain sums of the cubes by adding together the general law for consecutive values of $m$:

$$\begin{aligned}
&1 + (2+3+4) + (5+6+7+8+9) + \cdots + \big((N-1)^2+1) + \cdots + N^2\big)\\
&= (0+1) + (1+8) + (8+27) + \cdots + \big((N-1)^3 + N^3\big)
\end{aligned}$$

i.e.

$$\tfrac{1}{2}N^2(N^2+1) = 2(1^3 + 2^3 + \cdots + N^3) - N^3$$

i.e.

$$\tfrac{1}{2}N^2(N^2+2N+1) = 2(1^3 + 2^3 + \cdots + N^3)$$

as required.

**Question 3**

The indefinite integrals $I_1$ and $I_2$ are defined by

$$I_1 = \int \frac{\cos x}{\cos x + \sin x}\, \mathrm{d}x, \quad \text{and} \quad I_2 = \int \frac{\sin x}{\cos x + \sin x}\, \mathrm{d}x.$$

By considering $I_1 + I_2$ and $I_1 - I_2$, determine $I_1$ and $I_2$.

Determine similarly

$$\int \frac{\cos x}{a\cos x + b\sin x}\, \mathrm{d}x.$$

**Discussion**

This question has just one idea behind it, but what a good idea it is! (Not mine, alas.) It also works nicely for $I_1 = e^x \operatorname{sech} x$ and $I_2 = e^{-x} \operatorname{sech} x$, where $\operatorname{sech} x$ is the hyperbolic function $2(e^x + e^{-x})^{-1}$ (try it), but I cannot think of many other pairs of integrals for which it is useful.

For the second part, you need to make a small extension of the basic idea, thereby showing that you have understood it. Of course, you will want to check that your solution to the last part is consistent with the first part (by setting $a$=$b$=1). It is a good idea to check other special cases, such as $a = 0$ or $b = 0$. It is *always* a good idea, for any problem, to check special cases not only to test your result but also for the extra insight it can sometimes give.

Although the integrals in the question look as if they might be difficult to evaluate without the above trick, they can be tackled by more straightforward methods. For example, you could write the denominator in the form $R\cos(x - \alpha)$, change variable to $y = x - \alpha$, and then expand the resulting trig. function with appears in the numerator. It is worth trying this to see how the two parts of the answer arise.

Note that if a range of integration were given, it should not include the values of $x$ for which the denominator of the integrand is zero. If it did, the integral would be meaningless.

**Solution to Question 3**

We have

$$I_1 + I_2 = \int \frac{\cos x + \sin x}{\cos x + \sin x} \mathrm{d}x = \int 1 \, \mathrm{d}x = x + \text{ constant },$$

and

$$I_1 - I_2 = \int \frac{\cos x - \sin x}{\cos x + \sin x} \mathrm{d}x = \ln\big(\cos x + \sin x\big) + \text{ constant },$$

since the numerator of the second integrand is exactly the derivative of the denominator. Solving these two equations simultaneously for $I_1$ and $I_2$ gives

$$I_1 = \tfrac{1}{2}x + \tfrac{1}{2}\ln\big(\cos x + \sin x\big) + \text{ constant}$$

and

$$I_2 = \tfrac{1}{2}x - \tfrac{1}{2}\ln\big(\cos x + \sin x\big) + \text{ constant} .$$

For the second part, we use exactly the same principle. Setting

$$I_3 = \int \frac{\cos x}{a\cos x + b\sin x} \, \mathrm{d}x \quad \text{and} \quad I_4 = \int \frac{\sin x}{a\cos x + b\sin x} \, \mathrm{d}x,$$

we have

$$aI_3 + bI_4 = \int \frac{a\cos x + b\sin x}{a\cos x + b\sin x} \mathrm{d}x = \int 1 \, \mathrm{d}x = x + \text{ constant },$$

and

$$bI_3 - aI_4 = \int \frac{b\cos x - a\sin x}{a\cos x + b\sin x} \mathrm{d}x = \ln\big(a\cos x + b\sin x\big) + \text{ constant },$$

since again the numerator of the second integrand is exactly the derivative of the denominator. Thus

$$(a^2 + b^2) I_3 = ax + b\ln\big(a\cos x + b\sin x\big) + \text{ constant} .$$

**Question 4**

Prove that:

(i) if $a + 2b + 3c = 7x$, then

$$a^2 + b^2 + c^2 = (x - a)^2 + (2x - b)^2 + (3x - c)^2; \qquad (*)$$

(ii) if $2a + 3b + 3c = 11x$, then

$$a^2 + b^2 + c^2 = (2x - a)^2 + (3x - b)^2 + (3x - c)^2. \qquad (**)$$

State and prove a general result of which (i) and (ii) are special cases.

**Discussion**

The first two parts involve relatively straightforward algebra. To find the general rule requires an understanding of the algebra you have just done: why does it work in the way it does?

Of course, there are many other 'general rules' besides the one given in the solution. The situation is similar to the 'find the next number in the sequence' questions which come up in IQ tests. As no lesser figure than the philosopher Wittgenstein has pointed out, there is no correct answer. Given any finite sequence of numbers, a formula can always be found which will fit all the given numbers and which makes the next number (e.g.) 42.

Nevertheless, when the above question was set, almost everyone produced the same general result (or no result at all). Mathematicians seem to know what sort of thing is required, whereas electronic computers would not have the faintest idea what to do, though they would no doubt complete the algebra of the first two parts very rapidly.

**Solution to Question 4**

(i) The right hand side of the equation labelled $(*)$ on the previous page minus the left hand side of $(*)$ is

$$(x-a)^2+(2x-b)^2+(3x-c)^2-(a^2+b^2+c^2).$$

Expanding the squares and collecting up powers of $x$ reduces this to

$$(1+4+9)x^2-2x(a+2b+3c),$$

the terms without $x$ cancelling identically. The two remaining terms sum to zero, as required, on using the given equality $a+2b+3c=7x$ in the second term.

(ii) Proceeding as above using equation $(**)$ on the previous page gives

$$(2^2+3^2+3^2)x^2-2x(2a+3b+3c)$$

which again sums to zero, since now $2a+3b+3c=11x$.

For the general result, notice that the equations $(*)$ and $(**)$ differ only in the numerical coefficients of $x$ in the right hand sides. In both cases, these coefficients are the same as those occurring in the corresponding conditions ('if $a+2b+3c=7x$, ...').

We therefore investigate the statement:

If $\alpha a+\beta b+\gamma c=Ax$, then

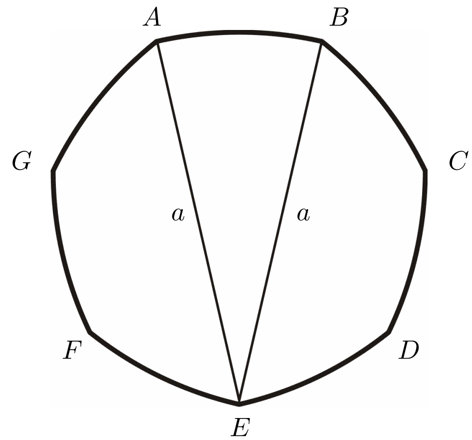
$$(\alpha x-a)^2+(\beta x-b)^2+(\gamma x-c)^2=(a^2+b^2+c^2).$$

The terms independent of $x$ will still cancel (we could attempt a generalisation of these coefficients as well). Expanding as above, we find that this formula holds if

$$A=\tfrac{1}{2}(\alpha^2+\beta^2+\gamma^2),$$

which is the expected generalisation.

**Question 5**




The diagram shows a British 50 pence coin. The seven arcs $AB$, $BC$, ... , $FG$, $GA$ are of equal length and each arc is formed from the circle of radius $a$ having its centre at the vertex diametrically opposite the mid-point of the arc. Show that the area of the face of the coin is

$$\frac{a^2}{2}\left(\pi - 7\tan\frac{\pi}{14}\right).$$

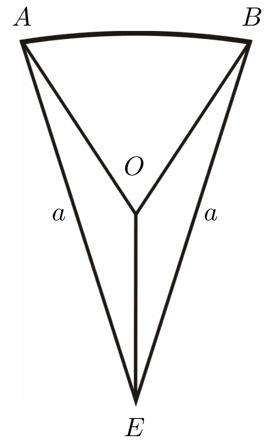
**Discussion**

The first difficulty with this elegant problem is drawing the diagram. However, you can simplify both the drawing and the solution by restricting your attention to just one sector of a circle of radius $a$.

The figure sketched above has *constant diameter*; it can roll between two parallel lines without losing contact with either. (This looks plausible and you can verify it by sellotaping some 50 pence coins together, but a solid proof is not very easy.) The distance between these lines is the diameter of the figure. Like a circle, the 50 pence piece has circumference equal to $\pi$ times the diameter, which is in fact always true for a figure with constant diameter.

As mathematicians, you might well want to generalise the above formula to the area of an $n$-sided coin (straightforward) and check that as $n \to \infty$, the formula gives the correct answer for a circle of diameter $a$.

**Solution to Question 5**




In the figure, the point $O$ is equidistant from each of three vertices $A$, $B$ and $E$. The plan is to find the area of the sector $AOB$ by calculating the area of $AEB$ and subtracting the areas of the two congruent isosceles triangles $OBE$ and $OAE$. The required area is 7 times this.

First we need angle $\angle AEB$. We know that $\angle AOB = 2\pi/7$ and hence $\angle BOE = \frac{1}{2}(2\pi - 2\pi/7) = 6\pi/7$ (using the sum of angles round a point). Finally,

$$\angle AEB = 2\angle OEB = 2 \times \tfrac{1}{2}(\pi - 6\pi/7) = \pi/7,$$

using the sum of angles of an isosceles triangle.

Now $ABE$ is a sector of a circle of radius $a$, so its area is

$$\pi a^2 \times \frac{\pi/7}{2\pi} = \frac{\pi a^2}{14}.$$

The area of triangle $OBE$ is $\frac{1}{2}BE\times$ height, i.e.

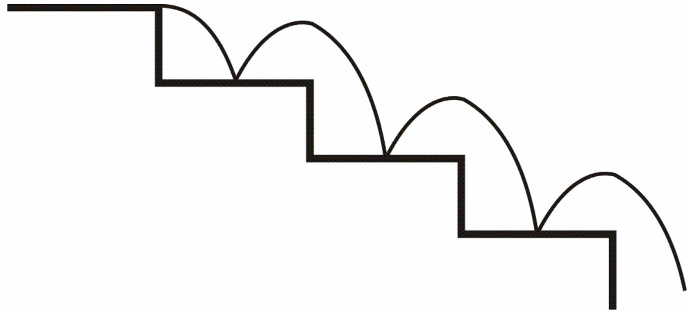
$$\tfrac{1}{2}a \times \frac{a}{2}\tan\angle OBE = \frac{a^2}{4}\tan\frac{\pi}{14}.$$

The area of the coin is therefore

$$7 \times \left(\frac{\pi a^2}{14} - 2 \times \frac{a^2}{4}\tan\frac{\pi}{14}\right),$$

which reduces to the given answer.

**Question 6**



A small ball is placed at the edge of the top step of a flight of steps. Each step has identical dimensions. The ball is kicked horizontally perpendicular to the edge of the top step (see diagram). On its first and second bounces it lands exactly in the middle of each of the first and second steps down from the top. Find the coefficient of restitution between the ball and the first step.

The ball continues bouncing down the steps hitting the middle of each successive step. What is the coefficient of restitution between the ball and the remaining steps?

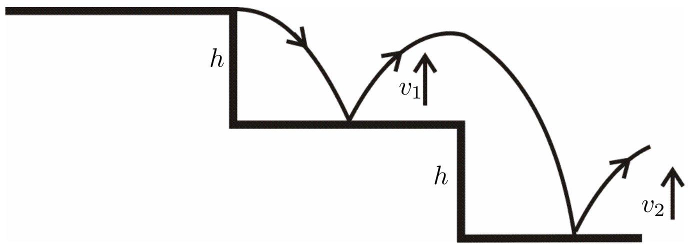
**Discussion**

This just involves application of the usual formulae for particle motion: '$s = ut - \frac{1}{2}gt^2$', '$v = u - gt$', etc. You also need to know Newton's law for bounces, which says that the velocity component normal to the fixed surface changes by a factor of $-$e, where e is the coefficient of restitution.

Although the results are independent of the dimensions of the steps (because if you scale each dimension by the same factor and scale the original horizontal velocity by the same factor, the time of each bounce, and hence also the vertical velocities, will scale with this factor; the coefficient of restitution is defined as the ratio of velocities, which is unchanged by the scaling), you will need to introduce some dimensions to handle the equations.

It is pleasing, and, at first sight, rather surprising that the ball can bounce in the middle of each step. This can be understood in terms of energy: the kinetic energy gained in falling down the steps is exactly balanced by the energy lost (in the form of heat and sound) during the inelastic impacts.

**Solution to Question 6**




Let the time to the first bounce be $T$. Then $h = \frac{1}{2}gT^2$, where $h$ is the height of the first (and every other) step. We have therefore

$$gT^2 = 2h. \qquad (*)$$

The vertical velocity downwards immediately before the first bounce is $gT$, so the vertical velocity upwards immediately after the first bounce, call it $v_1$, is given by

$$v_1 = \mathrm{e}gT, \qquad (**)$$

where e is the coefficient of restitution.

The horizontal velocity is not changed, either in the flight or in the process of bouncing. Since the horizontal distance between the first bounce to the second bounce is twice that travelled before the first bounce, the time between the first bounce and the second bounce is $2T$. Using '$s = ut - \frac{1}{2}gt^2$' for this period gives

$$\begin{aligned} -h &= v_1(2T) - \tfrac{1}{2}g(2T)^2 \\ &= (\mathrm{e}gT)(2T) - \tfrac{1}{2}g(2T)^2. \end{aligned}$$

Using $(*)$, this simplifies to $\mathrm{e} = 3/4$.

In order for the ball to bounce in the middle of each step, each bounce must be identical to the second bounce. This will happen if the vertical velocities after impact are the same (since the horizontal velocity remains constant). Let the vertical velocity just after the next bounce be $v_2$, so that the vertical velocity just before this bounce is $\tilde{\mathrm{e}}^{-1}v_2$, where $\tilde{\mathrm{e}}$ is the coefficient of restitution between the ball and the remaining steps. We have

$$\begin{aligned} (\tilde{\mathrm{e}}^{-1}v_2)^2 &= v_1^2 + 2gh && \text{using '}v^2 = u^2 + 2as\text{'} \\ &= \big((3/4)gT\big)^2 + 2gh && \text{using } \mathrm{e} = 3/4 \text{ in } (**) \\ &= (9/16)(2gh) + 2gh && \text{using } (*) \\ &= (25/16)(2gh). \end{aligned}$$

Thus $\tilde{\mathrm{e}}^{-1}v_2 = 5/4\sqrt{(2gh)}$. We want $v_2 = v_1$ and we have $v_1 = (3/4)\sqrt{(2gh)}$, so $\tilde{\mathrm{e}} = 3/5$.

**Question 7**

Frosty the snowman is made from two uniform spherical snowballs, of radii $2R$ and $3R$. The smaller (which is his head) stands on top of the larger. As each snowball melts, its volume decreases at a rate which is directly proportional to its surface area, the constant of proportionality being the same for each snowball. During melting, the snowballs remain spherical and uniform. When Frosty is half his initial height, show that the ratio of his volume to his initial volume is $37 : 224$.

Let $V$ and $h$ denote Frosty's total volume and height, respectively, at time $t$. Show that, for $2R < h \leqslant 10R$,

$$\frac{\mathrm{d}V}{\mathrm{d}h} = \frac{\pi}{8}(h^2 + 4R^2),$$

and derive the corresponding expression for $0 \leqslant h < 2R$.

Sketch $\mathrm{d}V/\mathrm{d}h$ as a function of $h$, for $4R \geqslant h \geqslant 0$. Hence give a rough sketch of $V$ as a function of $h$.

**Discussion**

This question is rather longer than the original STEP question; in fact, the first part is really sufficient for examination purposes.

The first part involves setting up an equation (the simplest possible differential equation, as it turns out) which gives the radii of the snowballs as a function of time. This is not difficult in itself, but it is necessary to think what variables you want to use and then go through a number of steps in the dark, without any reassurance from the question.

For the second part, you have to first recognise why there are two time periods which must be considered separately. Note that the inequalities at $h = 2R$ are strict; the function jumps at this value of $h$. This is the important feature which should appear on your first sketch. For the second sketch, you should just try to represent the area under the first sketch, rather than attempt a sketch of the cubic functions.

**Solution to Question 7**

For either snowball, the rate of change of volume $\mathrm{d}v/\mathrm{d}t$ at any time is related to the surface area $a$ by

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -ka, \qquad (*)$$

where $k$ is a positive constant. For a sphere of radius $r$, this becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}(4\pi r^3/3) = -k(4\pi r^2).$$

We can write $\mathrm{d}(r^3)/\mathrm{d}t$ as $3r^2\mathrm{d}r/\mathrm{d}t$, so equation $(*)$ is equivalent to (cancelling the factor of $4\pi r^2$)

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -k.$$

Thus $r = -kt + C$, where $C$ is a constant of integration.

Initially, Frosty's head has radius $2R$ and his body has radius $3R$, so the equations for the radii of the head and body at time $t$ are respectively

$$r = -kt + 2R \qquad \text{and} \qquad r = -kt + 3R.$$

Frosty's height $h$ is twice the sum of these radii, i.e. $h = 2(-2kt + 5R)$, which falls to half its original value of $10R$ when $t = 5R/(4k)$. At this time, the radii of the head and body are $3R/4$ and $7R/4$, so the ratio of his volume to his initial volume is

$$\frac{(4\pi/3)\,(3R/4)^3 + (4\pi/3)\,(7R/4)^3}{(4\pi/3)\,(2R)^3 + (4\pi/3)\,(3R)^3} = \frac{(3/4)^3 + (7/4)^3}{2^3 + 3^3} = \frac{37}{224}.$$

Before the head has completely melted, i.e. for $0 \leqslant t < 2R/k$, we have

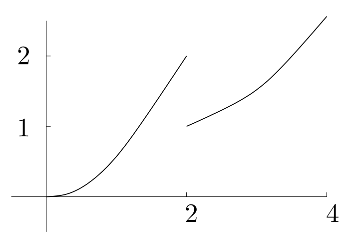
$$h = 2(-kt + 2R) + 2(-kt + 3R) \qquad \text{and} \qquad V = (4\pi/3)\left[(-kt + 2R)^3 + (-kt + 3R)^3\right],$$

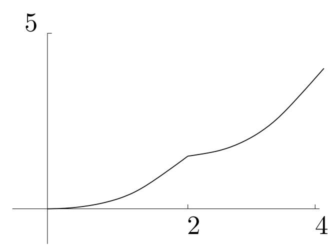
so

$$kt = 5R/2 - h/4 \quad \text{and} \quad V = (4\pi/3)\left[(h/4 - R/2)^3 + (h/4 + R/2)^3\right]$$

which gives the required expression for $\mathrm{d}V/\mathrm{d}h$. (Differentiate first, then simplify!) After the head has gone, i.e. for $2R/k \leqslant t < 3R/k$, there is just one snowball, so $V = (4\pi/3)(h/2)^3 = \pi h^3/6$ and $\mathrm{d}V/\mathrm{d}h = \pi h^2/2$.

In these sketches, the horizontal axes measure $h$ in units of $R$ and the vertical axes measure $\mathrm{d}V/\mathrm{d}h$ and $V$ in units of $\pi R^2$ and $\pi R^3$ respectively. The second sketch represents the area under the first sketch.



**Question 8**

Each day, I choose at random between my brown trousers, my grey trousers and my expensive but fashionable designer jeans. Also in my wardrobe, I have a black silk tie, a rather smart brown and fawn polka-dot tie, my regimental tie, and an elegant powder-blue cravat which I was given for Christmas. With my brown or grey trousers, I choose ties (including the cravat) at random, except of course that I don't wear the cravat with the brown trousers or the polka-dot tie with the grey trousers. With the jeans, the choice depends on whether it is Sunday or one of the six weekdays: on weekdays, half the time I wear a cream-coloured sweat-shirt with $E = mc^2$ on the front and no tie; otherwise, and on Sundays (when naturally I always wear a tie), I just pick at random from my four ties.

This morning, I received through the post a compromising photograph of myself. I often receive such photographs and they are equally likely to have been taken on any day of the week. However, in this particular photograph, I am wearing my black silk tie. Show that, on the basis of this information, the probability that the photograph was taken on Sunday is 11/68.

I should have mentioned that on Mondays I lecture on calculus and I therefore always wear my jeans (to make the lectures seem easier to understand). Find, on the basis of the complete information, the probability that the photograph was taken on Sunday.

**Discussion**

This is hardly a probability question at all, more an exercise in translating information to a diagram, so you should not be put off if you have not done much work on probability. The most straightforward way to tackle it is to use a tree-diagram, but you could equally well use Bayes' Theorem in the form

$$\mathrm{P}(A|B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)},$$

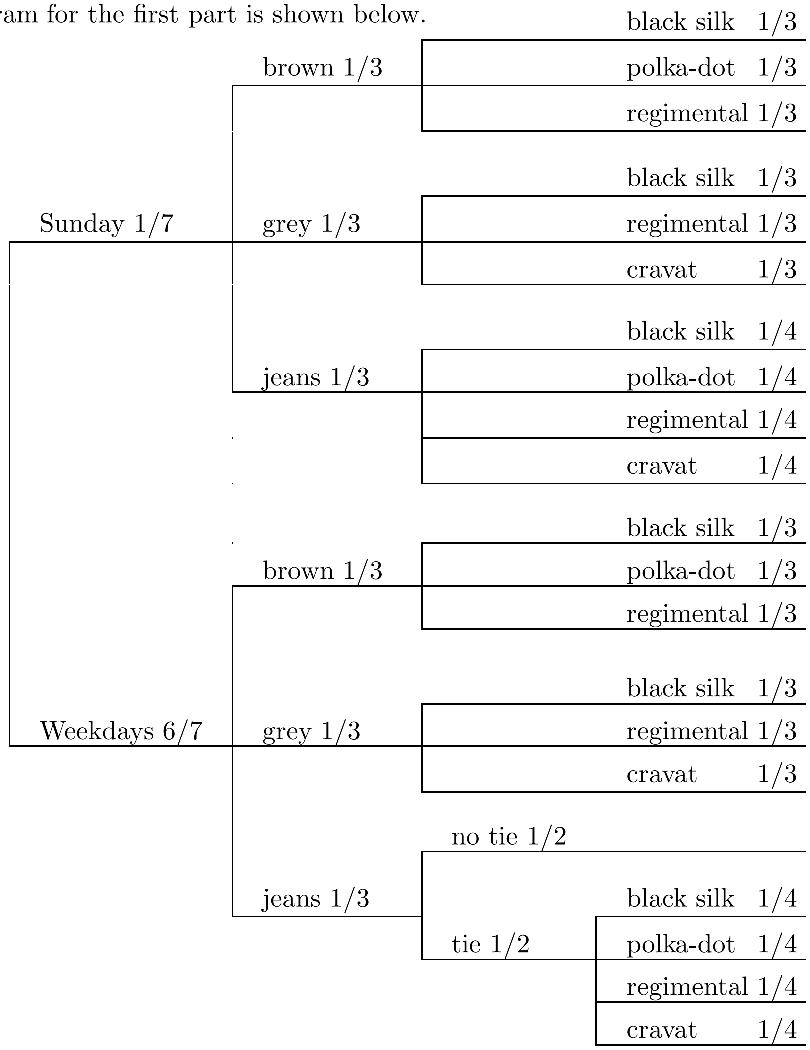
as in Question 13.

The phrase 'at random' here means 'with equal probability'.

I have seen an $E = mc^2$ sweat-shirt, with a picture of Einstein on the back, but I didn't find out where it came from. However, if you like this sort of thing, you can get (or at least you used to be able to get) a Fermat's Last Theorem t-shirt from the Isaac Newton Institute in Cambridge (01223 335999), which is where the proof of the theorem was announced. There turned out to be a gap in the proof, but this has since been bridged. The proof is not given on the t-shirt: it is a truly wonderful proof, but the t-shirt is too small to contain it. (It runs to several hundred pages.)

**Solution to Question 8**

The tree diagram for the first part is shown below.




The probability that I am wearing my black silk tie is

$$\tfrac{1}{7}(\tfrac{1}{3}\times\tfrac{1}{3}+\tfrac{1}{3}\times\tfrac{1}{3}+\tfrac{1}{3}\times\tfrac{1}{4})+\tfrac{6}{7}(\tfrac{1}{3}\times\tfrac{1}{3}+\tfrac{1}{3}\times\tfrac{1}{3}+\tfrac{1}{3}\times\tfrac{1}{2}\times\tfrac{1}{4})=\tfrac{22}{504}+\tfrac{114}{504}.$$

the first bracket being the probability of my wearing the black silk tie on Sunday. The probability that it is Sunday, given that I am wearing my black silk tie, is therefore (cancelling factors of 504)

$$\tfrac{22}{22+114}=\tfrac{11}{68}.$$

For the second part, I need to consider Mondays separately and add an extra branch to the tree. The new probability that I am wearing my black tie is (Monday represented by the last bracket)

$$\tfrac{1}{7}(\tfrac{1}{3}\times\tfrac{1}{3}+\tfrac{1}{3}\times\tfrac{1}{3}+\tfrac{1}{3}\times\tfrac{1}{4})+\tfrac{5}{7}(\tfrac{1}{3}\times\tfrac{1}{3}+\tfrac{1}{3}\times\tfrac{1}{3}+\tfrac{1}{3}\times\tfrac{1}{2}\times\tfrac{1}{4})+\tfrac{1}{7}(\tfrac{1}{2}\times\tfrac{1}{4})=\tfrac{22}{504}+\tfrac{95}{504}+\tfrac{9}{504}.$$

The probability that it is Sunday, given that I am wearing my black silk tie, is now (cancelling factors of 504)

$$\tfrac{22}{22+95+9}=\tfrac{11}{63}.$$

**Question 9**

A chocolate orange consists of a sphere of delicious smooth uniform chocolate of mass $M$ and radius $a$, sliced into segments by planes through a fixed axis. It stands on a horizontal table with this axis vertical and it is held together by a narrow ribbon round its equator. Show that the tension in the ribbon is at least $\frac{3}{32}Mg$.

[You may assume that the centre of mass of a segment of angle $2\theta$ is at distance $\dfrac{3\pi a \sin\theta}{16\theta}$ from the axis.]

**Discussion**

This question can be done by the usual methods (resolving forces and taking moments about a suitably chosen point). Since the chocolate is smooth, there is no friction. The ribbon may be elastic, so it could be tighter than is needed just to keep the orange together. At the lowest tension possible the orange is on the point of falling apart, so there are no forces between the faces of the segments, except at the point of contact with the table.

Intuitively, one would expect this minimum tension to be independent of the number of segments ($> 1$), so to check the answer, you could just do the case of two equal segments, thereby obtaining the given answer quite rapidly.

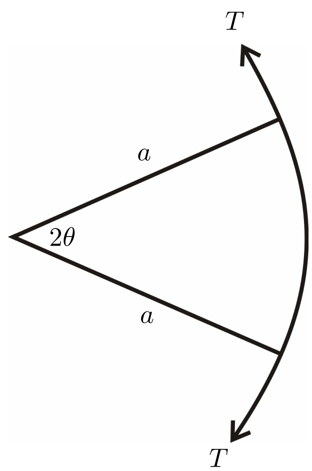
The ribbon is said to be thin, but actually this has to be interpreted as massless as well as thin; a ribbon with mass would simply drop off the equator of the orange.

Another way of tackling this sort of problem is to use the principle of virtual work, for which you imagine that the system relaxes a very small amount (in this case, by allowing the ribbon to stretch) and equate the work done against the constraints (here, tension times extension) to the change of potential energy of the system. In many cases, this method is much simpler, but here it turns out to be very difficult: not recommended at all.

When the question was first set in 1985, the formula for the distance of the centre of mass of the segment was incorrectly given, and the answer for the tension was also incorrect, but consistent. Not surprisingly, no one pointed it out, either at the time or afterwards. With a bit of luck, it is correct now.

When I set this question originally, I was rewarded with a parcel from a well-known manufacturer of high quality chocolate confection; not a parcel as large as I had hoped for (perhaps they spotted the error), but better than nothing. It seemed worth including the question in this booklet in case they wanted an opportunity to make amends . . . .

**Solution to Question 9**



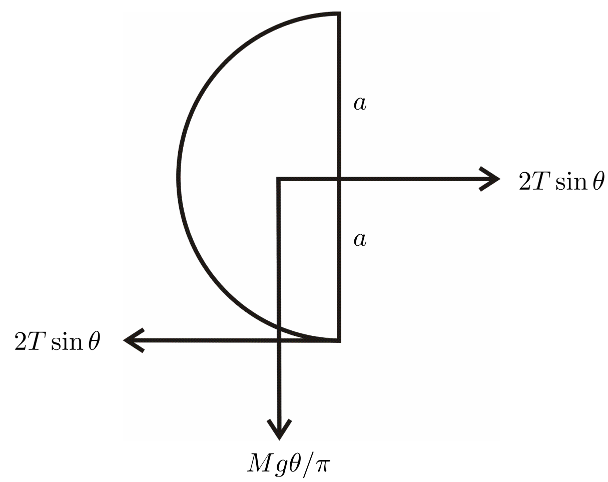

The above diagram shows a horizontal cross-section of a segment of the orange.

Resolving the tension $T$ in the ribbon as shown gives a horizontal component of force due to tension on the segment of

$$2T\sin\theta$$

towards the axis.

The volume of the segment is a fraction $2\theta/(2\pi)$ of the volume of the sphere, so the segment has mass $M(2\theta)/(2\pi)$. The weight of the segment gives a force of $Mg\,\theta/\pi$ acting downwards through the centre of mass as shown in the diagram below, which is a vertical cross-section of a segment.




Taking moments about the point of contact of the table and the segment gives

$$(Mg\theta/\pi) \times 3\pi a\sin\theta/(16\theta) = 2T\sin\theta \times a$$

which gives the required answer.

**Question 10**

Explain the geometrical relationship between the points in the Argand diagram represented by the complex numbers $z$ and $a + (z - a)e^{i\theta}$.

(i) Write down the necessary and sufficient conditions that the distinct complex numbers $\alpha$, $\beta$ and $\gamma$ represent the vertices of an equilateral triangle taken in clockwise order. Show that $\alpha$, $\beta$ and $\gamma$ represent the vertices of an equilateral triangle if and only if

$$\alpha^2 + \beta^2 + \gamma^2 - \beta\gamma - \gamma\alpha - \alpha\beta = 0. \qquad (\dagger)$$

(ii) Find necessary and sufficient conditions on the complex numbers $a$, $b$ and $c$ for the roots of the equation

$$z^3 + az^2 + bz + c = 0$$

to lie at the vertices of an equilateral triangle in the Argand diagram.

**Discussion**

This exercise in the geometric interpretation of the Argand diagram also required careful handling of 'necessary and sufficient'. The very last part can be done just as easily without using the first parts.

Jean Argand (1768–1822) was a Parisian bookkeeper. His single contribution to mathematics was the invention and elaboration of a geometric representation of complex numbers, but it was enough to ensure his place in the history of mathematics.

In fact, his discovery was anticipated 10 years previously by the Norwegian Caspar Wessel. We do not speak of the 'Wessel diagram' because Wessel's work remained unknown for a further 100 years, while Argand published a book about his own work. The book contained various applications, including a proof of the fundamental theorem of algebra (about the existence of zeros of any polynomial, i.e. about the existence of roots of polynomial equations) and a demonstration of the geometrical meaning of the identity

$$\cos m\theta + i \sin m\theta = (\cos\theta + i\sin\theta)^m$$

discovered a century earlier by de Moivre.

Argand also joined in the debate about which functions of the variable $x + iy$ can be written in the form $a + ib$; he believed, incorrectly, that $i^{\,i}$ could not be expressed in this form. Although $i^{\,i}$ looks strange, it can easily be simplified to produce a rather surprising result:

$$i^{\,i} = (e^{i\pi/2})^i = e^{-\pi/2}\,.$$

We should have anticipated that $i^{\,i}$ is real (maybe you did): if we take the complex conjugate, we find

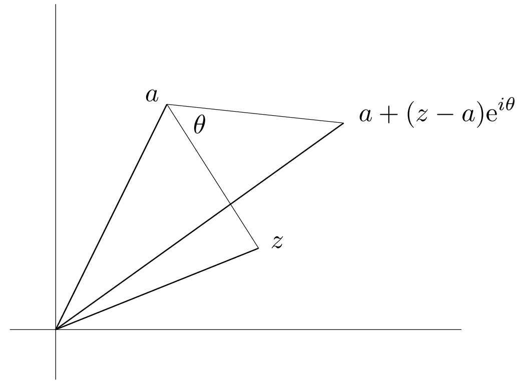
$$\left(i^{\,i}\right)^* = (-i)^{-i} = \left(\frac{1}{i}\right)^{-i} = i^{\,i}\,.$$

**Solution to Question 10**

The point represented by

$$a + (z - a)\mathrm{e}^{i\theta}$$

is a rotation by $\theta$ anticlockwise about the point represented by $a$ of the point represented by $z$.




There are a number of ways we could write the required conditions. One way (suggested by the first part of the question) is

$$(\beta - \alpha) = (\gamma - \alpha)\mathrm{e}^{\pi i/3},$$

which is both necessary and sufficient. (NB: the conditions $|\alpha - \beta| = |\beta - \gamma| = |\gamma - \alpha|$ are necessary, but not sufficient because they do not distinguish between clockwise and anticlockwise ordering.)

For $\alpha$, $\beta$ and $\gamma$ to form an equilateral triangle in either clockwise or anticlockwise order, either

$$(\beta - \alpha) - (\gamma - \alpha)\mathrm{e}^{\pi i/3} = 0$$

or

$$(\beta - \alpha) - (\gamma - \alpha)\mathrm{e}^{-\pi i/3} = 0.$$

This is equivalent to saying

$$\Big[(\beta - \alpha) - (\gamma - \alpha)\mathrm{e}^{\pi i/3}\Big]\Big[(\beta - \alpha) - (\gamma - \alpha)\mathrm{e}^{-\pi i/3}\Big] = 0. \qquad (*)$$

Multiplying this out, remembering that $\mathrm{e}^{-\pi i/3} + \mathrm{e}^{\pi i/3} = 1$, gives the required condition. The condition is necessary and sufficient: it is clearly necessary, because if the triangle is equilateral, then one of the brackets in $(*)$ vanishes (depending on whether the points are labelled clockwise or anti-clockwise); it is sufficient, because if (†) overleaf holds, then we can factorise the left hand side to give $(*)$ and, because the product of the factors vanishes, one of the factors must vanish.

For the last part, let the the roots of the cubic be $\alpha$, $\beta$ and $\gamma$. Then

$$a = -(\alpha + \beta + \gamma) \qquad \text{and} \qquad b = \beta\gamma + \gamma\alpha + \alpha\beta.$$

The roots lie at the vertices of an equilateral triangle if and only if the condition (†) overleaf, i.e. if and only if

$$(-a)^2 - 3b = 0,$$

which is the required answer.

Note that we could have done the last part of the question more directly. Setting $b = a^2/3$ in the cubic gives

$$z^3 + az^2 + \tfrac{1}{3}a^2 z + c = 0 \quad \text{i.e.} \quad (z + \tfrac{1}{3}a)^3 + c = a^3/27.$$

The roots of this equation are $z = -\frac{1}{3}a + c_1,\ -\frac{1}{3}a + c_2,\ -\frac{1}{3}a + c_3$, where $c_1$, $c_2$ and $c_3$ are the three cube roots of $a^3/27 - c$; these roots clearly lie at the vertices of an equilateral triangle (since $c_1$, $c_2$ and $c_3$ do). Conversely, if the roots lie at the vertices of an equilateral triangle with centre $d$, then $(z - d)^3 = k$, for some $k$. Multiplying out shows that $a = 3d$ and $b = 3d^2$, and hence $a^2 = 3b$.

**Question 11**

Show by means of the substitution $x = \alpha\cos^2\theta + \beta\sin^2\theta$, or otherwise, that

$$\int_\alpha^\beta \frac{1}{\sqrt{(x-\alpha)(\beta-x)}}\,\mathrm{d}x = \pi$$

when $\alpha < \beta$. What is the value of the integral when $\beta < \alpha$?

Using the substitution $t = x^{-1}$, show that

$$\int_a^b \frac{1}{t\sqrt{(t-a)(b-t)}}\,\mathrm{d}t = \frac{\pi}{\sqrt{ab}}$$

when $0 < a < b$.

Evaluate, for $0 < c < d$, the integral

$$\int_c^d \frac{1}{u\sqrt{(u^2-c^2)(d^2-u^2)}}\,\mathrm{d}u.$$

**Discussion**

The substitution suggested in the first paragraph looks complicated but actually produces a remarkable simplification of the integral. You have to be a little careful with the limits. Note that you do not have to perform more calculations to evaluate the integral in the case $\beta < \alpha$: the value can be inferred from the previous case.

In the last paragraph, you have to think of a suitable substitution.

The normal way to do the first integral would be to write the denominator in the form

$$\sqrt{-\alpha\beta + (\alpha+\beta)x - x^2} \equiv \sqrt{\tfrac{1}{4}(\beta-\alpha)^2 - \big(x - \tfrac{1}{2}(\alpha+\beta)\big)^2}$$

and use the substitution $x - \frac{1}{2}(\alpha+\beta) = \frac{1}{2}(\beta-\alpha)\sin\phi$. The standard double angle trig. identities show that the relation between this substitution and that suggested is $\phi = 2\theta - \pi/2$.

Note that $t = 0$ is not in the range of integration of the second integral, since $a > 0$. This is necessary. Near $t = 0$ the integrand is approximately $(t\sqrt{-ab}\,)^{-1}$ whose integral involves $\ln t$. This 'blows up' at $t = 0$, so the integral is not defined; not permitted, in fact. You might think that the first integral is similarly bad because of the zeros in the denominator at $x = \alpha$ and $x = \beta$. However, near $x = \alpha$, the integrand is approximately $(x-\alpha)^{-\frac{1}{2}}(\beta-\alpha)^{-\frac{1}{2}}$ whose integral, $2(x-\alpha)^{+\frac{1}{2}}(\beta-\alpha)^{-\frac{1}{2}}$, is well-behaved at $x = \alpha$. In general, it is OK to have a zero of the form $t^k$ in the denominator provided $k < 1$.

**Solution to Question 11**

Making the suggested change of variable, we have

$$\begin{aligned}(x-\alpha)(\beta-x) &= \big(\alpha(\cos^2\theta-1)+\beta\sin^2\theta\big)\big(\beta(1-\sin^2\theta)-\alpha\cos^2\theta\big)\\ &= \big(-\alpha\sin^2\theta+\beta\sin^2\theta\big)\big(\beta\cos^2\theta-\alpha\cos^2\theta\big)\\ &= (\beta-\alpha)^2\sin^2\theta\cos^2\theta\end{aligned}$$

and

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = -2\alpha\cos\theta\sin\theta+2\beta\sin\theta\cos\theta = 2(\beta-\alpha)\sin\theta\cos\theta. \tag{$*$}$$

Now we need to work out the limits of the transformed integral. When $x=\alpha$,

$$\alpha = \alpha\cos^2\theta+\beta\sin^2\theta$$

i.e.

$$(\alpha-\beta)\sin^2\theta = 0.$$

Ignoring the possibility $\alpha=\beta$ for which the integral is trivially zero, we must have $\sin^2\theta=0$, or $\theta=n\pi$ where $n$ is an integer. Since the original transformation only defines $\theta$ up to an addition of $n\pi$ (i.e, adding $n\pi$ on to $\theta$ does not change the value of $x$), we can choose to take $\theta=0$ as the lower limit of the integral.

When $x=\beta$, we find similarly that $\theta=\frac{1}{2}\pi+n\pi$. This time, we are not free to choose $n$. It is a pretty safe guess that we must take $\theta=\frac{1}{2}\pi$. We can justify this by noting that we want $x$ to increase from $\alpha$ to $\beta$, and if we take any other value for the upper limit, this would not happen.

The substitution therefore gives

$$\int_\alpha^\beta \frac{1}{\sqrt{(x-\alpha)(\beta-x)}}\,\mathrm{d}x = 2\int_0^{\frac{1}{2}\pi}\frac{1}{\sqrt{(\beta-\alpha)^2\sin^2\theta\cos^2\theta}}\;(\beta-\alpha)\sin\theta\cos\theta\,\mathrm{d}\theta = 2\int_0^{\frac{1}{2}\pi}\mathrm{d}\theta = \pi. \tag{$**$}$$

If $\beta<\alpha$, the square root in $(**)$ becomes $-(\beta-\alpha)\sin\theta\cos\theta$, following the convention that the root with the positive value is required. This introduces an additional minus sign, so the answer is $-\pi$. Note that when $\beta<\alpha$, equation $(*)$ implies that $x$ decreases as $\theta$ increases from 0 to $\pi/2$, which is what we want.

Setting $t=x^{-1}$ in the second integral gives

$$\int_a^b \frac{1}{t\sqrt{(t-a)(b-t)}}\,\mathrm{d}t = \int_{1/a}^{1/b}\frac{x}{\sqrt{(x^{-1}-a)(b-x^{-1})}}\;\frac{-1}{x^2}\,\mathrm{d}x$$

$$= -\int_{1/a}^{1/b}\frac{1}{\sqrt{(1-xa)(xb-1)}}\,\mathrm{d}x = -\int_{1/a}^{1/b}\frac{1}{\sqrt{(ab)(x-a^{-1})(b^{-1}-x)}}\,\mathrm{d}x = \frac{\pi}{\sqrt{ab}}.$$

For the very last step, we have used the previous result, noting that $b^{-1}<a^{-1}$.

A sensible try for the substitution in the last integral would be $u^2=t$, which transforms the integral into the previous one, with $a$ and $b$ replaced by $c^2$ and $d^2$:

$$\int_c^d \frac{1}{u\sqrt{(u^2-c^2)(d^2-u^2)}}\,\mathrm{d}u = \int_{c^2}^{d^2}\frac{1}{t^{\frac{1}{2}}\sqrt{(t-c^2)(d^2-t)}}\,\frac{1}{2t^{\frac{1}{2}}}\,\mathrm{d}t = \frac{\pi}{2cd}.$$

**Question 12**

Prove that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

Show how the cubic equation

$$24x^3 - 72x^2 + 66x - 19 = 0 \qquad (\dagger)$$

can be reduced to the form

$$4z^3 - 3z = k$$

by means of the substitutions $y = x + a$ and $z = by$, for suitable values of the constants $a$ and $b$. Hence find the three roots of equation ($\dagger$) expressing your answers in the form $p + q\cos r$.

Show, by means of a counterexample, or otherwise, that not all cubic equations of the form

$$x^3 + \alpha x^2 + \beta x + \gamma = 0$$

can be solved by this method.

**Discussion**

This method of solving cubic equations is due to François Viéta (1540–1603), the French mathematician who is regarded as the father of modern algebra. He developed much of the language of algebra that is familiar to us today. Previously, most algebraic expressions were written out in words, the unknown being referred to as 'the thing' (in Latin).

The long-sought solution of the general cubic was found, in 1535 by Niccolò Tartaglia (c. 1500-57). This achievement brought him great celebrity, and he spent the next 10 years visiting the crowned heads of Europe and solving their cubics for them. However, he was persuaded to divulge his secret, on the promise of complete confidentiality, by Girolamo Cardano (1501–76), who promptly published it in his algebra book *The Great Art*. There followed an acrimonious dispute between Tartaglia and Cardano which preoccupied much of Tartaglia's later life. At one point, the two agreed to a public duel, in which they would each bring along their favourite mathematical problems for the other to solve. However, it never took place.

The method derived above, based on the trisection of an angle, is restricted (as hinted in the last paragraph of the question) to certain types of cubic, but can be readily adapted to all other cubics. The solution to the last paragraph, and further discussion, is appended below (there was no room for it overleaf) so if you have not tried the question, stop reading now and try it.

There are various counterexamples we could think of. Clearly, the cubic $x^3 = 1$ cannot be reduced to the form $4z^3 - 3z = k$ by means of the given transformations: any attempt to generate the $-3z$ term will always give us a quadratic term.

However, this is a rather trivial counterexample (though it would get us full marks), because we can very easily solve this equation. In fact, it is easy to see that any equation can be reduced to the forms $4z^3 - 3z = k$, $4z^3 + 3z = k$ or $4z^3 = k$, where $k$ is real. (The ratio of the coefficient of the linear and cubic terms involves $b^2$, as in equation ($*$) overleaf, so we cannot change the sign of this ratio without making $b$ imaginary, which would make $k$ imaginary.) We can only solve the first of these equations by the trig method if $|k| \leqslant 1$, since it has to equal $\cos 3\theta$.

However, if $|k| > 1$, we can use the same method with cos replaced by cosh. Moreover, we can solve the second of these forms using the triple-angle formula for $\sinh 3\theta$. You can easily obtain these other formulae without further work. For example, to obtain the cosh formula, set $\theta = i\phi$ in the trig triple-angle formula and use $\cos i\phi = \cosh\phi$. (To prove this last result, just express everything in terms of exponentials.) You can obtain the sin formula by the transformation $\theta = \pi/2 - \psi$, from which the sinh formula follows by setting $\psi = i\phi$ and using $\sin i\phi = i\sinh\phi$.

**Solution to Question 12**

To prove the trig. identity, we can use the formulae for $\cos(a+b)$, etc:

$$\cos 3\theta = \cos(2\theta + \theta) = \cos 2\theta \cos\theta - \sin 2\theta \sin\theta$$

and then use double angle formulae:

$$= (\cos^2\theta - \sin^2\theta)\cos\theta - (2\sin\theta\cos\theta)\sin\theta = \cos^3\theta - 3\sin^2\theta\cos\theta,$$

which turns into the given answer on setting $\sin^2\theta = 1 - \cos^2\theta$. When calculating this sort of trig identity, it is worth being aware of any symmetries which could give a check on the algebra. Here, $\cos 3\theta$ is unchanged when $\theta \to -\theta$, so the right hand sides must never contain $\sin\theta$ on its own (though $\sin^2\theta$ is permitted).

Alternatively, we can use the complex representation of cos. We have

$$4\cos^3\theta = 4\left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right)^3 = \tfrac{1}{2}\left(e^{3i\theta} + 3e^{i\theta} + 3e^{-i\theta} + e^{-3i\theta}\right) = \cos 3\theta + 3\cos\theta.$$

For the transformation of the cubic, first set $x = y - a$ and see what happens. The left hand side of (†) becomes

$$24(y-a)^3 - 72(y-a)^2 + 66(y-a) - 19 = 24y^3 - 72(a+1)y^2 + (72a^2 + 144a + 66)y - (24a^3 + 72a^2 + 66a + 19).$$

One of the aims of the transformation is to get rid of the quadratic term (which is absent from $4z^3 - 3z = k$), so we choose $a = -1$. The resulting cubic is

$$24y^3 - 6y - 1 = 0.$$

Now we set $y = z/b$:

$$(24/b^3)z^3 - (6/b)z - 1 = 0.$$

Our aim is to make the ratio of the first two coefficients $-4/3$, so we choose

$$\frac{24/b^3}{6/b} = 4/3 \quad \text{i.e.} \quad b = \sqrt{3}, \qquad (\text{or } b = -\sqrt{3} \text{ would do equally well})$$

giving

$$(24/3\sqrt{3})z^3 - (6/\sqrt{3})z - 1 = 0 \quad \text{i.e.} \quad 4z^3 - 3z = \sqrt{3}/2. \qquad (*)$$

We can solve this equation by means of the trig identity derived in the first paragraph. If $z = \cos\theta$, then the equation is equivalent to

$$\cos 3\theta = \sqrt{3}/2.$$

The cubic will have three roots, so we need the three values of $\theta$ which satisfy this equation and give different values of $\cos\theta$. These are given by

$$3\theta = \frac{\pi}{6}, \;\; 2\pi - \frac{\pi}{6}, \;\; 2\pi + \frac{\pi}{6}.$$

The solutions are therefore

$$z = \cos\frac{1}{18}\pi, \;\; \cos\frac{11}{18}\pi, \;\; \cos\frac{13}{18}\pi.$$

Unravelling the transformations gives $x = z/\sqrt{3} + 1$, so

$$x = 1 + \frac{1}{\sqrt{3}}\cos\frac{1}{18}\pi, \;\; 1 + \frac{1}{\sqrt{3}}\cos\frac{11}{18}\pi, \;\; 1 + \frac{1}{\sqrt{3}}\cos\frac{13}{18}\pi.$$

### Question 13

My two friends, who shall remain nameless, but whom I shall refer to as $P$ and $Q$, both told me this afternoon that there is a body in my fridge. I'm not sure what to make of this, because $P$ tells the truth with a probability of $p$, while $Q$ (independently) tells the truth with a probability of only $q$. I haven't looked in the fridge for some time, so if you had asked me this morning, I would have said that there was just as likely to be a body in the fridge as not. Clearly, in view of what my friends have told me, I must revise this estimate. Explain carefully why my new estimate of the probability of there being a body in the fridge should be

$$\frac{pq}{1-p-q+2pq}.$$

I have now been to look in the fridge and there is indeed a body in it; perhaps more than one. It seems to me that only my enemy $E_1$ or my other enemy $E_2$ or (with a bit of luck) both $E_1$ and $E_2$ could be in my fridge, and this evening I would have judged these three possibilities equally likely. But tonight I asked $P$ and $Q$ separately whether $E_1$ was in the fridge, and they each said that she was. What should be my new estimate of the probability that both $E_1$ and $E_2$ are in my fridge?

Of course, I always tell the truth.

### Discussion

The most difficult part of this problem is unravelling the narrative! The first paragraph says essentially 'what is the probability that there is body in the fridge, given that $P$ and $Q$ both say there is?'. It can therefore be tackled by the usual methods of conditional probability: tree diagrams, for example, or Bayes' theorem. All the other words in the first paragraph are there to tell you about the *a priori* probabilities of the events, without knowledge of which the question above is meaningless.

In the second paragraph, the situation becomes more complicated, but the method used for the first paragraph will still work. Note how much more difficult it is when the answer is not given; when the question was originally set, most candidates arrived at the given answer to the first part but were not sufficiently confident to extend their method to the second paragraph: they received 8/20 for their efforts.

The last paragraph is not entirely frivolous; if I may have lied about what my friends answered when I asked them if there is a body in the fridge, the problem becomes difficult. However, my claim to be truthful is vacuous (it tells you nothing) because I may be lying. (Contrast with the statement 'I am lying', which is inconsistent.)

**Solution to Question 13**

This problem can be solved using tree diagrams; see Question 8 for an example of this kind of solution. A more sophisticated (but not necessarily better) method is to use Bayes' theorem.

Bayes' theorem, in its simplest form, states:

$$\mathrm{P}(B|A) = \frac{\mathrm{P}(B) \times \mathrm{P}(A|B)}{\mathrm{P}(A)}.$$

Here, we take the events $A$ and $B$ to be

$$\begin{aligned} A &= P \text{ and } Q \text{ both say that there is a body in the fridge} \\ B &= \text{there is a body in the fridge} \end{aligned}$$

From the information given in the question, $P(B) = \frac{1}{2}$, so

$$\mathrm{P}(B|A) = \frac{\frac{1}{2} \times pq}{\mathrm{P}(A)}.$$

Now

$$\begin{aligned} \mathrm{P}(A) &= \mathrm{P}(\text{there is a body}) \times \mathrm{P}(P \text{ and } Q \text{ both say there is}) \\ &\qquad + \mathrm{P}(\text{there is not a body}) \times \mathrm{P}(\text{they both say there is}) \\ &= \tfrac{1}{2} \times pq + \tfrac{1}{2} \times (1-p)(1-q) \end{aligned}$$

which gives the required answer.

For the second paragraph, let

$$\begin{aligned} X &= P \text{ and } Q \text{ both say that } E_1 \text{ is in the fridge} \\ Y &= E_1 \text{ and } E_2 \text{ are in the fridge} \end{aligned}$$

There are three possibilities (since we know that there is at least one body in the fridge): only $E_1$ is in the fridge; only $E_2$ is in the fridge; and both $E_1$ and $E_2$ are in the fridge. These are given as equally likely, so the *a priori* probabilities are each 1/3.

We want $\mathrm{P}(Y|X)$, which by Bayes' theorem is

$$\frac{\mathrm{P}(Y) \times \mathrm{P}(X|Y)}{\mathrm{P}(X)} = \frac{\frac{1}{3} \times pq}{\mathrm{P}(X)}.$$

Now

$$\begin{aligned} \mathrm{P}(X) &= \mathrm{P}(\text{only } E_1 \text{ is in the fridge}) \times \mathrm{P}(P \text{ and } Q \text{ told the truth}) \\ &\qquad + \mathrm{P}(\text{both } E_1 \text{ and } E_2 \text{ are in the fridge}) \times \mathrm{P}(P \text{ and } Q \text{ both told the truth }) \\ &\qquad\quad + \mathrm{P}(\text{only } E_2 \text{ is in the fridge}) \times \mathrm{P}(P \text{ and } Q \text{ both lied }) \\ &= \tfrac{1}{3} \times pq + \tfrac{1}{3} \times pq + \tfrac{1}{3} \times (1-p)(1-q) \end{aligned}$$

so the answer is $pq/(1-p-q+3pq)$.

**Question 14**

Find the simultaneous solution of the three linear equations

$$\begin{aligned} a^2x + ay + z &= a^2 &\quad (1)\\ ax + y + bz &= 1 &\quad (2)\\ a^2bx + y + bz &= b &\quad (3) \end{aligned}$$

for all possible real values of $a$ and $b$.

**Discussion**

The simplest way to tackle this is to use an elimination (or substitution) method: attempt to isolate the unknowns by adding multiples of the equations. You have to be careful because for some values of the parameters $a$ and $b$, the quantity you want to divide by will be zero; such special cases must be treated separately. Do the general case first, noting the values of $a$ and $b$ for which there are problems, then go back and deal with the problems.

It is possible to solve this problem by matrix methods. Write the equations in the form $\mathbf{A}\mathbf{x} = \mathbf{c}$, where $\mathbf{c}$ is the column vector $(a^2, 1, b)^{\mathrm{T}}$, and then compute the inverse of the $3 \times 3$ matrix $\mathbf{A}$. This involves a good deal of algebra, and the special cases, where the matrix $\mathbf{A}$ is singular ($\det \mathbf{A} = 0$), need careful thought.

Rather than try to invert the matrix, as discussed above, you can use the method of Gaussian elimination, which is just a fancy title for a systematic method of elimination; in terms of matrices, it just involves subtracting multiples of rows of $\mathbf{A}$ from other rows to achieve an upper triangular matrix. Provided the same operations are performed on the right hand side of the equation (i.e. to the rows of the column matrix $\mathbf{c}$), no damage is done to the equations — they are different, but they still have the same solutions. If you are familiar with this technique, you might like to try it.

**Solution to Question 14**

We can isolate $z$ (i.e. eliminate $x$ and $y$) in one go because of the special form of the equations: (1) $-\, a \times$ (2) gives

$$\begin{aligned} & z(1-ab) = a^2 - a \\ \Longrightarrow \qquad & z = \frac{a(a-1)}{1-ab} \qquad\qquad \text{provided } ab \neq 1. \end{aligned}$$

Note the condition $ab \neq 1$; we have to take into account 'all possible values of $a$ and $b$'.

The structure of the original equations encourages three separate eliminations, rather than one elimination followed by back substitution. Taking (2) $-$ (3) gives

$$\begin{aligned} & x(a - a^2 b) = 1 - b \\ \Longrightarrow \qquad & x = \frac{1-b}{a(1-ab)} \qquad\qquad \text{provided } ab \neq 1 \text{ and } a \neq 0. \end{aligned}$$

Similarly, (3) $-\, b \times$ (1) gives

$$\begin{aligned} & y(1-ab) = b - a^2 b \\ \Longrightarrow \qquad & y = \frac{b(1-a^2)}{1-ab} \qquad\qquad \text{provided } ab \neq 1. \end{aligned}$$

We have now found the general solution, which is valid except when either (i) $ab = 1$ or (ii) $a = 0$. It is probably best to return to the original equations to investigate these special cases.

Case (i). If $ab = 1$, we can set $b = a^{-1}$ in equations (1), (2) and (3):

$$a^2 x + ay + z = a^2, \quad ax + y + a^{-1} z = 1, \quad ax + y + a^{-1} z = a^{-1}.$$

The second and third of these can only agree if $a = 1$; otherwise the equations are inconsistent and there are no solutions. If $a = 1$, the three equations are all equivalent to

$$x + y + z = 1.$$

Any three numbers which sum to 1 satisfy the equations. Geometrically, this represents a whole plane of solutions.

Case (ii). If $a = 0$, equations (1), (2) and (3) become

$$z = 0, \quad y + bz = 1, \quad y + bz = b \quad \text{i.e.} \quad z = 0, \quad y = 1, \quad y = b.$$

These equations do not determine $x$ and are inconsistent (and have no solutions) unless $b = 1$. Geometrically, the solution is represented by any point on the line

$$\mathbf{x} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} .$$

**Question 15**

Give rough sketches of the function $\tan^k \theta$ for $0 \leqslant \theta \leqslant \frac{1}{4}\pi$ in the two cases $k = 1$ and $k \gg 1$.

Show that for any positive integer $n$

$$\int_0^{\pi/4} \tan^{2n+1}\theta \; \mathrm{d}\theta = (-1)^n \left( \tfrac{1}{2}\ln 2 + \sum_{m=1}^{n} \frac{(-1)^m}{2m} \right), \tag{†}$$

and deduce that

$$\ln 2 = -\sum_{m=1}^{\infty} \frac{(-1)^m}{m}. \tag{‡}$$

Show similarly that

$$\frac{\pi}{4} = -\sum_{m=1}^{\infty} \frac{(-1)^m}{2m-1}.$$

**Discussion**

The first part of the question tells you what to do (in not-very-easy stages) and the second part of the question tests your understanding of what you have done and why you have done it by asking you to apply the method to a different but essentially similar problem.

In the first paragraph, you have to see how the function $\tan^k\theta$ changes when $k$ becomes very big; you need only a rough sketch to show that you have understood the important point. This should be done (at least for a first attempt) by thought, not by means of a calculator. Clearly, the conclusion of this first part has to be used in later parts of the question.

If you are stuck with the integral of the second paragraph, you might like to think in terms of a recurrence formula, i.e. a formula relating $I_{2n+1}$ and $I_{2n-1}$ (in the obvious notation).

The last paragraph is closely related to the previous result; there is a clue to the difference in the denominators of the fraction in the two sums.

The series derived above for $\pi/4$ is usually called Leibniz' formula, although the general series for $\tan^{-1} x$ was written down by Gregory in 1671, two years before Leibniz. It was one of the first explicit formulae for $\pi$ (there had previously been a formula involving an infinite product). Previously, the value of $\pi$ could only be estimated geometrically, by (for example) approximating the circumference of a circle by the edges of an inscribed regular polygon. (Using a square, for example, gives $\pi \approx 2\sqrt{2}$.)

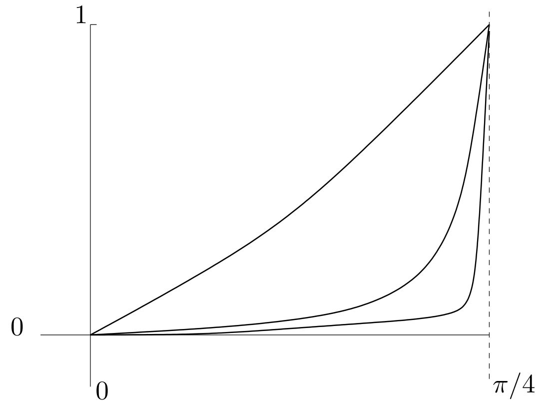
You might think that the method of obtaining Leibniz' formula suggested in this question is rather indirect; one could instead just integrate the formula

$$\frac{\mathrm{d}\tan^{-1} x}{\mathrm{d}x} = \frac{1}{1+x^2} = 1 - x^2 + x^4 - \cdots$$

term by term and get the result immediately by setting $x = 1$. The virtue of the method used in the question is that it gives an explicit form (an integral) of the remainder after $n$ terms of the series. We were able to show, by means of a sketch, that the remainder tends to zero as $n$ tends to infinity; in other words, we showed that the series converges. Although the sketch method of proof is a bit crude, it can easily be made more rigorous once the concept of integration is more carefully defined.

**Solution to Question 15**

For $0 \leqslant \tan\theta < \frac{1}{4}\pi$, we have $\tan\theta < 1$, so that $\tan^k\theta$ is close to zero (i.e. much smaller than 1) when $k$ is large. This is illustrated in the figure which shows three cases: for $k=1$ the graph is mildly curved; for larger $k$ the graph hugs the $x$-axis. The graphs all pass through the point $(\pi/4, 1)$.




To evaluate the integral, let

$$I_{2n+1} = \int_0^{\pi/4} \tan^{2n+1}\theta \ \mathrm{d}\theta. \qquad (*)$$

We shall express $I_{2n+1}$ in terms of $I_{2n-1}$ using the relation $\tan^2\theta = \sec^2\theta - 1$:

$$I_{2n+1} = \int_0^{\pi/4} \tan^{2n-1}\theta\,(-1+\sec^2\theta)\ \mathrm{d}\theta = -I_{2n-1} + \int_0^1 u^{2n-1}\,\mathrm{d}u = -I_{2n-1} + \frac{1}{2n}.$$

To evaluate the second integral, we have set $u = \tan\theta$ so that $\mathrm{d}u = \sec^2\theta\,\mathrm{d}\theta$ (though you could leave out this step and evaluate the integral directly).

Repeating the process gives

$$I_{2n+1} = -I_{2n-1} + \frac{1}{2n} = I_{2n-3} - \frac{1}{2(n-1)} + \frac{1}{2n} = \cdots = (-1)^n I_1 + \frac{1}{2n} - \frac{1}{2(n-2)} + \cdots + (-1)^{n+1}\frac{1}{2}.$$

The above sum (starting with $1/(2n)$) is the same as that in (†) overleaf, so it only remains to evaluate $I_1$, corresponding to $n=0$ in $(*)$:

$$I_1 = \int_0^{\pi/4} \tan\theta\ \mathrm{d}\theta = -\ln(\cos\theta)\Big|_0^{\pi/4} = -\ln(1/\sqrt{2}) = \tfrac{1}{2}\ln 2,$$

as required.

We deduce the expression (‡) overleaf for $\ln 2$ using the first part of the question. When $n$ is very large, $I_{2n+1}$ is very small, being the area under a graph which is almost zero for almost all of the range of integration. In the limit $n\to\infty$, we set $I_{2n+1}=0$ in (†) which leads immediately to (‡).

To obtain the formula for $\pi/4$, we follow the above method using $I_{2n}$ instead of $I_{2n+1}$. This time we have to calculate $I_0$:

$$I_0 = \int_0^{\pi/4} 1\ \mathrm{d}\theta = \frac{\pi}{4}.$$

**Question 16**

Given that $x$, $y$ and $z$ satisfy

$$\begin{aligned} x^2 - yz &= a, &(1)\\ y^2 - zx &= b, &(2)\\ z^2 - xy &= c, &(3)\end{aligned}$$

where $a$, $b$ and $c$ are real, not all equal, and $a+b+c>0$, show that

$$b - c = (y-z)(x+y+z)\,. \tag{4}$$

By considering

$$(b-c)^2 + (c-a)^2 + (a-b)^2,$$

or otherwise, show that

$$x + y + z = \frac{\triangle}{a+b+c}, \tag{5}$$

where

$$\triangle^2 = a^3 + b^3 + c^3 - 3abc. \tag{6}$$

Hence solve equations (1), (2) and (3) for $x$, $y$ and $z$.

**Discussion**

This is a test of algebra. The question is well structured, so that you have to have just one insight at each stage. Note that the equations are symmetric: if you replace $b$ by $a$, $c$ by $b$ and $a$ by $c$, and perform the same operation with $x$, $y$ and $z$, the equations are unchanged. This leads to a saving of work, but can also be used to check for algebraic errors.

The method of solution along which you are steered is a bit convoluted; it is therefore necessary to satisfy yourself that the method does actually lead to solutions of the original equations and that you have not missed any on the way. This is best done by considering the relationship of the equations you actually solved to the original equations.

Note that the sign of $\triangle$ is not determined. If $x$, $y$ and $z$ satisfy the equations, then so also do $-x$, $-y$ and $-z$, and for this solution $\triangle$ will have the same magnitude but the opposite sign. It is not surprising that there are two solutions for given $a$, $b$ and $c$: the equations are quadratic.

Geometrically, equations (1) – (3) represent hyperboloids: for example, equation (1) can be written $x^2 + (y-z)^2/4 - (y+z)^2/4 = a$, which looks something like an eggtimer if $a > 0$ and like a pair of radar dishes if $a < 0$. Three eggtimers intersect in two points, corresponding to the solutions you are about to find, but the six radar dishes do not intersect at all; in this case, $a+b+c<0$ which means that $\triangle$ is imaginary.

There are neater ways of solving these equations (for example, using matrices, but not in an obvious way). When the question was set (without the intermediate steps) in a Cambridge University examination for undergraduates in January of 1860, the examiners provided a model solution. Their idea was to square equation (1) and subtract the product of equations (2) and (3); a good plan, but not one that many people would think of under examination conditions.

**Solution to Question 16**

To obtain equation (4), we just subtract equation (3) from equation (2) and factorise:

$$b-c=(y^2-zx)-(z^2-xy)=(y^2-z^2)+(xy-xz)=(y-z)(y+z)+(y-z)x=(y-z)(x+y+z)$$

By symmetry, we can write down equations for $c-a$ and $a-b$. Squaring and adding these three equations gives

$$(b-c)^2+(c-a)^2+(a-b)^2=2(x^2+y^2+z^2-yz-zx-xy)(x+y+z)^2=2(a+b+c)(x+y+z)^2 \quad (*)$$

the second equality coming from adding together equations (1), (2) and (3). Then we expand the left hand side of $(*)$:

$$(b-c)^2+(c-a)^2+(a-b)^2=2(a^2+b^2+c^2-bc-ca-ab)\equiv 2(a^3+b^3+c^3-3abc)/(a+b+c)\,.$$

You can check the equivalence by just multiplying it all out. Now cancelling the 2s and taking square roots gives equation (5).

To solve for $x$, $y$ and $z$, we go back to equation (4). We have

$$y=z+\frac{(b-c)(a+b+c)}{\triangle}$$

and similarly

$$x=z+\frac{(a-c)(a+b+c)}{\triangle}\,.$$

We could substitute these into the original equations to find $z$, but it is easier to use instead equation (5):

$$3z+\frac{(a+b-2c)(a+b+c)}{\triangle}=\frac{\triangle}{a+b+c}.$$

This, with slight rearrangement, gives a formula for $z$ and the corresponding results for $x$ and $y$ can be deduced by symmetry. Being mathematicians, you no doubt cannot resist the urge to tidy up:

$$\begin{aligned}3z &= \frac{\triangle}{a+b+c}-\frac{(a+b-2c)(a+b+c)}{\triangle}=\frac{\triangle^2-(a+b-2c)(a+b+c)^2}{(a+b+c)\triangle}\\ &= \frac{(a^2+b^2+c^2-bc-ca-ab)-(a+b-2c)(a+b+c)}{\triangle}\\ &= \frac{3c^2-3ab}{\triangle}\end{aligned}$$

and the 3 cancels. Note the pleasing likeness between the solution and the original equations.

Of course, we have to worry about dividing by $\triangle$: could $\triangle=0$? From

$$2\triangle\equiv\big((b-c)^2+(c-a)^2+(a-b)^2\big)\big(a+b+c\big)\,,$$

we see that $\triangle=0$ only if either $a=b=c$ or $a+b+c=0$, both of which are excluded in the question. If $a=b=c$, then it is easy to see that any $x$, $y$ and $z$ lying on the plane $x+y+z=0$ satisfy the equations. If $a+b+c<0$, we see from equation $(*)$ that the solutions are complex. If $a+b+c=0$, equation $(*)$ shows that $a=b=c$.

**Question 17**

If $y = \mathrm{f}(x)$, the inverse of f is given by *Lagrange's identity*:

$$\mathrm{f}^{-1}(y) = y + \sum_{1}^{\infty} \frac{1}{n!} \frac{\mathrm{d}^{n-1}}{\mathrm{d}y^{n-1}} \big[y - \mathrm{f}(y)\big]^{n},$$

when this series converges.

(i) Verify Lagrange's identity when $\mathrm{f}(x) = ax$.

(ii) Show that one root of the equation $x - \frac{1}{4}x^3 = \frac{3}{4}$ is

$$x = \sum_{0}^{\infty} \frac{3^{2n+1}(3n)!}{n!(2n+1)!4^{3n+1}}. \qquad (\dagger)$$

(iii) Find a solution for $x$, as a series in $\lambda$, of the equation $x = e^{\lambda x}$.

[You may assume that the series in part (ii) converges and that the series in parts (i) and (iii) converge for suitable $a$ and $\lambda$.]

**Discussion**

This looks pretty frightening at first, because of the complicated and unfamiliar formula. However, its bark is worse than its bite. Once you have decided what you need to find the inverse of, you just substitute it into the formula and see what happens. Do not worry about the use of the word 'convergence'; this can be ignored. It is just included to satisfy the legal eagles who will point out that the series might not have a finite sum (e.g. $1 + 2 + 3 + \cdots$).

In part (ii), you can solve the cubic by normal means (find one root by inspection, factorise and use the usual formula to solve the resulting quadratic equation). The root found by Lagrange's equation is the one closest to zero. If you solve by normal means, you will see that equation (†) turns out to be a very obscure way of writing a very familiar quantity.[1] Lagrange's method works essentially for any cubic, since transformations similar to those of Question 12 can be used to reduce the general cubic to a form amenable to Lagrange's formula, whereas the factorisation method only works if you happen to spot one factor.

In part (iii), you might like to sketch the two functions and see for what range of values of $\lambda$ the equation does have a root. This will correspond exactly to the range for which the series converges. (Compare with Question 29.)

Lagrange was one of the leading mathematicians of the 18th century; Napoleon referred to him as the 'lofty pyramid of the mathematical sciences'. He attacked a wide range of problems, from celestial mechanics to number theory. In the course of his investigation of the roots of polynomial equations, he discovered group theory (in particular, his eponymous theorem about the order of a subgroup dividing the order of the group), though the term 'group' and the systematic theory had to wait until Galois and Abel in the first part of the 19th century.

Lagrange's formula, produced before the advent of the theory of integration in the complex plane, which allows a relatively straightforward derivation, testifies to his remarkable mathematical ability. It is practically forgotten now, but in its day it had a great impact. The applications given above give an idea how important it was, in the age before computers.

[1] The expansion sums to 1; I don't know how you can see that directly. The formula for the real root of a cubic gives in this case the real part of $-\big(12 + 4i\sqrt{13/27}\,\big)^{\frac{1}{3}}$, but expanding this binomially does not result in (†). Note that $(\,1 + i\sqrt{13/3})^3 = -12 - 4i\sqrt{13/27}$.

**Solution to Question 17**

(i) The inverse of $\mathrm{f}(x) = ax$ is $\mathrm{f}^{-1}(y) = y/a$, since then $\mathrm{f}^{-1}\big(\mathrm{f}(x)\big) = x$ as required. Substituting into Lagrange's formula gives

$$\begin{aligned} \mathrm{f}^{-1}(y) = y + \sum_1^\infty \frac{1}{n!}\,\frac{\mathrm{d}^{n-1}}{\mathrm{d}y^{n-1}}[y - ay]^n &= y + \sum_1^\infty (1-a)^n \frac{1}{n!}\,\frac{\mathrm{d}^{n-1}}{\mathrm{d}y^{n-1}} y^n \\ &= y + \sum_1^\infty (1-a)^n y \\ &= y + y\frac{(1-a)}{1-(1-a)}, \end{aligned}$$

where for the last equality we have summed the geometric progression. This simplifies to $y/a$, thus verifying Lagrange's formula.

[The geometric progression converges provided $|1-a| < 1$, i.e. if $0 < a < 2$.]

(ii) Let $\mathrm{f}(x) = x - \frac{1}{4}x^3$. Then the equation becomes $\mathrm{f}(x) = \frac{3}{4}$, so we must find $\mathrm{f}^{-1}(\frac{3}{4})$. Again, we just substitute into Lagrange's formula, (leaving the $y$ arbitrary, for the moment)

$$\begin{aligned} \mathrm{f}^{-1}(y) = y + \sum_1^\infty \frac{1}{n!}\,\frac{\mathrm{d}^{n-1}}{\mathrm{d}y^{n-1}}[y - (y - \tfrac{1}{4}y^3)]^n &= y + \sum_1^\infty \frac{1}{n!}\,\frac{\mathrm{d}^{n-1}}{\mathrm{d}y^{n-1}}[\tfrac{1}{4}y^3]^n \\ &= y + \sum_1^\infty \frac{1}{4^n n!}\,\frac{\mathrm{d}^{n-1}}{\mathrm{d}y^{n-1}} y^{3n} \\ &= y + \sum_1^\infty \frac{1}{4^n n!}\,\frac{(3n)!}{(2n+1)!} y^{2n+1}. \end{aligned}$$

Now we just set $y = \frac{3}{4}$ to obtain the given result. Note that we have in fact found a solution to the equation

$$x^3 - 4x + 4y = 0,$$

which, for general values of $y$, cannot be solved by the factorisation method mentioned in the discussion overleaf. (Though there is a formula for the general solution of a cubic equation.)

[Using the approximation $n! \approx (n/e)^2$, which is a simplified version of Stirling's formula, it can be seen that the series converges provided $|y| < 4/\sqrt{27}$. If we require $x$ to lie between the two turning points of $\mathrm{f}(x)$, then $\mathrm{f}^{-1}(y)$ will be well defined; this gives $|x| < 2/\sqrt{3}$ which translates into the above condition on $y$.]

(iii) The obvious choice for f is $\mathrm{f}(x) = x - e^{\lambda x}$, in which case the equation becomes $\mathrm{f}(x) = 0$ and we require $\mathrm{f}^{-1}(0)$. Again substituting into Lagrange's formula gives

$$\begin{aligned} \mathrm{f}^{-1}(y) = y + \sum_1^\infty \frac{1}{n!}\,\frac{\mathrm{d}^{n-1}}{\mathrm{d}y^{n-1}}\left[e^{\lambda y}\right]^n &= y + \sum_1^\infty \frac{1}{n!}\,\frac{\mathrm{d}^{n-1}}{\mathrm{d}y^{n-1}} e^{ny\lambda} \\ &= y + \sum_1^\infty \frac{1}{n!}\,(n\lambda)^{n-1} e^{ny\lambda}. \end{aligned}$$

Setting $y = 0$ gives a series for the root:

$$\sum_1^\infty \frac{n^{n-1}}{n!}\,\lambda^{n-1},$$

which cannot be further simplified.

[This series converges for $|\lambda| < \mathrm{e}^{-1}$.]

**Question 18**

Write down the general solution of the equation

$$\ddot{x} = -k^2 x,$$

where $k$ is a constant.

A truck is towing a trailer of mass $m$ across level ground by means of an elastic tow-rope of natural length $l$ and modulus of elasticity $\lambda$. At first the rope is slack and the trailer is stationary. The truck then accelerates until the rope becomes taut and thereafter the truck travels in a straight line at constant speed $u$. Assuming that the effect of friction on the trailer is negligible, show that the trailer will collide with the truck at time

$$\frac{\pi}{k} + \frac{l}{u}$$

after the rope first becomes taut, where $k = \sqrt{\lambda/(lm)}$.

**Discussion**

No element of this problem is difficult, but some thought is required since no intermediate steps are given. You will need to choose variables (there are various possibilities, including the position of the trailer, the extension of the rope or the position of the trailer relative to the truck).

You will reach an equation of motion for the trailer involving the tension in the rope ($\lambda\times$extension$/l$) which will boil down to the simple harmonic equation if you choose the right variable, or the simple harmonic equation with a term linear in $t$ if you do not. In the first case, you can just write down the general solution, while in the second case you will have to work out the particular integral.

**Solution to Question 18**

The general solution can be written in the form

$$x = A\cos kt + B\sin kt\,,$$

where $A$ and $B$ are arbitrary constants.

At time $t$ after the rope becomes taut, let $y$ be the displacement of the trailer from its original position and $x$ be the extension of the tow-rope. Then at $t = 0$, we have $y = \dot{y} = x = 0$.

The displacement of the truck from its position when the rope became taut is $ut$, so at time $t$

$$ut = x + y. \tag{$*$}$$

The tension in the rope is $\lambda x/l$ so the equation of motion of the trailer while the rope is under tension is

$$m\ddot{y} = \frac{\lambda x}{l}.$$

Differentiating equation $(*)$ twice gives $\ddot{y} = -\ddot{x}$, so

$$-m\ddot{x} = \frac{\lambda x}{l}$$

and the solution to this equation is

$$x = A\cos kt + B\sin kt, \qquad \text{or} \quad y = ut - A\cos kt - B\sin kt.$$

Since $y = \dot{y} = 0$ at $t = 0$, we have $A = 0$ and $B = u/k$.

The extension of the rope is $(u/k)\sin kt$, so the rope becomes slack again when $kt = \pi$. At this time, the speed of the trailer is

$$\dot{y} = u - Bk\cos\pi = 2u$$

and its speed relative to the truck is $u$. The further time required to travel the extra distance $l$ is $l/u$. The truck and trailer therefore collide after time $\pi/k + l/u$.

**Question 19**

Let $\mathbf{A}$ and $\mathbf{C}$ be $n \times n$ matrices, and denote the $n \times n$ unit matrix by $\mathbf{I}$ and the $n \times n$ zero matrix by $\mathbf{O}$.

(i) Show that if $\mathbf{C}$ is non-singular then $(\mathbf{C}^{\mathrm{T}})^{-1} = (\mathbf{C}^{-1})^{\mathrm{T}}$, where $\mathbf{C}^{\mathrm{T}}$ denotes the transpose of $\mathbf{C}$.

(ii) Now assume that the matrix $(\mathbf{I} - \mathbf{A})$ is non-singular and let $\mathbf{B} = (\mathbf{I} + \mathbf{A})(\mathbf{I} - \mathbf{A})^{-1}$. Show that $\mathbf{B}^{\mathrm{T}}\mathbf{B} = \mathbf{I}$ if and only if $\mathbf{A}^{\mathrm{T}} + \mathbf{A} = \mathbf{O}$.

[You may assume any elementary properties of matrices that you need, including the results $(\mathbf{PQ})^{\mathrm{T}} = \mathbf{Q}^{\mathrm{T}}\mathbf{P}^{\mathrm{T}}$ and $(\mathbf{P} + \mathbf{Q})^{\mathrm{T}} = \mathbf{P}^{\mathrm{T}} + \mathbf{Q}^{\mathrm{T}}$, which hold for any $n \times n$ matrices $\mathbf{P}$ and $\mathbf{Q}$.]

**Discussion**

This is really only for those who have gone beyond most Further Mathematics Syllabuses in the area of matrices. In the 1970s, it was thought appropriate to teach matrices at GCSE.[2] That meant that more advanced matrices could be taught at A-level, which is why this question found its way onto a STEP paper.

You may be tempted to prove these results for $2 \times 2$ matrices. In general, this is not a bad plan. You may get some insight into why the results hold which would allow you to formulate a proof in the general case. You should resist the temptation here: you probably won't get any insight but you probably will get a mess.

In real life, you can often get a handle on a proof by looking at simple cases, but in STEP exams the examiners would very likely have directed you towards the special case if it could have been helpful. Here, the question says uncompromisingly '$n \times n$' to steer you away from unhelpful specialisation.

Remember the important distinctions between matrices and ordinary numbers. Matrix multiplication does not commute ($\mathbf{AB} \neq \mathbf{BA}$, in general) and inverses of matrices do not always exist (which means that cancellation, corresponding to multiplication by inverses, is not always possible). Inverses in fact exist only for non-singular matrices, i.e. those with non-zero determinant. Note that the notation

$$\frac{\mathbf{A}}{\mathbf{B}}$$

is not used for $\mathbf{AB}^{-1}$ because it is ambiguous: it does not distinguish between left and right multiplication by the inverse matrix. Note also that, as in ordinary algebra, $(\mathbf{I} + \mathbf{A})^{-1} \neq \mathbf{I} + \mathbf{A}^{-1}$, in general.

To prove the first part, you have to work from the definition of the inverse: it is the matrix which you have to multiply by (either on the left or on the right) to obtain the unit matrix.

For the second part, be careful of the 'if and only if'; if you are not sure that you have proved both 'necessary' and 'sufficient', then back up a bit and continue the proof in two strands. In the first strand, assume that $\mathbf{B}^{\mathrm{T}}\mathbf{B} = \mathbf{I}$ and prove that $\mathbf{A}^{\mathrm{T}} + \mathbf{A} = \mathbf{O}$. In the second strand assume that $\mathbf{A}^{\mathrm{T}} + \mathbf{A} = \mathbf{O}$ and prove that $\mathbf{B}^{\mathrm{T}}\mathbf{B} = \mathbf{I}$.

[2] A completely batty idea, in my humble opinion, and one that was quite damaging: the time take to learn to minipulate matrices without really having much idea what they were meant that basic mathematical skills were marginalised.

**Solution to Question 19**

(i) We need to show that $(\mathbf{C}^{-1})^{\mathrm{T}}\mathbf{C}^{\mathrm{T}} = \mathbf{I}$, because multiplying this on the right by $(\mathbf{C}^{\mathrm{T}})^{-1}$ gives the required formula. Start from $\mathbf{C}\mathbf{C}^{-1} = \mathbf{I}$ and take the transpose, using the given formula for the transpose of a product:

$$\begin{aligned} \mathbf{C}\mathbf{C}^{-1} = \mathbf{I} \quad &\Longrightarrow \quad (\mathbf{C}\mathbf{C}^{-1})^{\mathrm{T}} = \mathbf{I}^{\mathrm{T}} \\ &\Longrightarrow \quad (\mathbf{C}^{-1})^{\mathrm{T}}\mathbf{C}^{\mathrm{T}} = \mathbf{I} \end{aligned}$$

which is the required result. For the second line, we have used the fact that $\mathbf{I}$ is symmetric (since all non-diagonal entries are zero).

(ii) We are told that $\mathbf{B} = (\mathbf{I}+\mathbf{A})(\mathbf{I}-\mathbf{A})^{-1}$ so

$$\begin{aligned} \mathbf{B}^{\mathrm{T}} &= \left[(\mathbf{I}-\mathbf{A})^{-1}\right]^{\mathrm{T}}(\mathbf{I}+\mathbf{A})^{\mathrm{T}} && (1) \\ &= \left[(\mathbf{I}-\mathbf{A})^{\mathrm{T}}\right]^{-1}(\mathbf{I}+\mathbf{A})^{\mathrm{T}} && (2) \\ &= \left[\mathbf{I}-\mathbf{A}^{\mathrm{T}}\right]^{-1}(\mathbf{I}+\mathbf{A}^{\mathrm{T}}) && (3) \end{aligned}$$

using the formula for the transpose of a product in equation (1) and the formula proved in part (i) for equation (2). For equation (3), we have used the fact that the transpose of a sum of matrices is the sum of the transposes of the matrices (i.e. $(\mathbf{C}+\mathbf{D})^{\mathrm{T}} = \mathbf{C}^{\mathrm{T}} + \mathbf{D}^{\mathrm{T}}$).

Therefore,

$$\begin{aligned} \mathbf{B}^{\mathrm{T}}\mathbf{B} &= (\mathbf{I}-\mathbf{A}^{\mathrm{T}})^{-1}(\mathbf{I}+\mathbf{A}^{\mathrm{T}})(\mathbf{I}+\mathbf{A})(\mathbf{I}-\mathbf{A})^{-1} \\ &= (\mathbf{I}-\mathbf{A}^{\mathrm{T}})^{-1}(\mathbf{I}+\mathbf{A}^{\mathrm{T}}+\mathbf{A}+\mathbf{A}^{\mathrm{T}}\mathbf{A})(\mathbf{I}-\mathbf{A})^{-1}. && (4) \end{aligned}$$

(Be careful of the ordering here; $\mathbf{A}^{\mathrm{T}}\mathbf{A}$ is not the same as $\mathbf{A}\mathbf{A}^{\mathrm{T}}$.)

If $\mathbf{B}^{\mathrm{T}}\mathbf{B} = \mathbf{I}$, we can multiply equation (4) on the left by $(\mathbf{I}-\mathbf{A}^{\mathrm{T}})$ and on the right by $(\mathbf{I}-\mathbf{A})$ to obtain

$$(\mathbf{I}-\mathbf{A}^{\mathrm{T}})(\mathbf{I}-\mathbf{A}) = \mathbf{I}+\mathbf{A}^{\mathrm{T}}+\mathbf{A}+\mathbf{A}^{\mathrm{T}}\mathbf{A}$$

i.e.

$$\mathbf{I}-\mathbf{A}^{\mathrm{T}}-\mathbf{A}+\mathbf{A}^{\mathrm{T}}\mathbf{A} = \mathbf{I}+\mathbf{A}^{\mathrm{T}}+\mathbf{A}+\mathbf{A}^{\mathrm{T}}\mathbf{A}.$$

This can only hold if $\mathbf{A}^{\mathrm{T}}+\mathbf{A} = \mathbf{O}$, as required.

Conversely, if $\mathbf{A}^{\mathrm{T}}+\mathbf{A} = \mathbf{O}$, we can replace $\mathbf{A}^{\mathrm{T}}$ by $-\mathbf{A}$ everywhere in equation (4):

$$\begin{aligned} \mathbf{B}^{\mathrm{T}}\mathbf{B} &= (\mathbf{I}+\mathbf{A})^{-1}(\mathbf{I}-\mathbf{A}^2)(\mathbf{I}-\mathbf{A})^{-1} \\ &= (\mathbf{I}+\mathbf{A})^{-1}(\mathbf{I}+\mathbf{A})(\mathbf{I}-\mathbf{A})(\mathbf{I}-\mathbf{A})^{-1} \\ &= \mathbf{I}, \end{aligned}$$

as required.

**Question 20**

Let $a$, $b$, $c$, $d$, $p$ and $q$ be positive integers. Prove that

(i) If $a < b$ and $d < c$, then $bc - ad \geqslant a + c$;

(ii) If $\dfrac{a}{b} < p < \dfrac{c}{d}$, then $(bc - ad)p \geqslant a + c$;

(iii) If $\dfrac{a}{b} < \dfrac{p}{q} < \dfrac{c}{d}$, then $p \geqslant \dfrac{a + c}{bc - ad}$ and $q \geqslant \dfrac{b + d}{bc - ad}$ .

Find all fractions with denominators less that 20 which lie between 8/9 and 9/10.

**Discussion**

It is important to realise that this question is about integers; the results would not be true if $a$, $b$, ... were allowed to be non-integers. (You can convince yourself of this by replacing $c$ and $d$ in part (iii) by $kc$ and $kd$, and making $k$ very small. This would not affect the 'if' statement, but the 'then' statement would imply the $p$ and $q$ both had to be very large.)

The essential point to notice is that the 'if's have strict inequalities and the 'then's allow equality. For integers, this is a big difference.

You might wonder why anyone could have wanted this peculiar result, other than to torture examination candidates with. One answer lies in the series of fractions called Farey series. A Farey series is the set of all (lowest order) fractions with both numerators and denominators less than a given integer $N$, arranged in order of magnitude. For $N = 5$, the fractions are

$$\frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{1}{1}, \frac{2}{5}, \frac{2}{3}, \frac{3}{5}, \frac{3}{4}, \frac{4}{5},$$

which, arranged in order of magnitude, give the Farey series

$$\frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1}.$$

The results proved above can be used in the theory of Farey series, which is itself of general importance in number theory. For example, for any consecutive fractions $a/b$, $p/q$ and $c/d$ in a Farey series, the inequalities proved in part (iii) are in fact equalities. Thus, given the $(n-1)$th and $(n+1)$th terms, you can find the $n$th term. However, there is an annoying cancellation if you try to solve the equations to find the $(n+1)$ term from the $n$th and $(n-1)$th terms (try it: you just get $bp - aq = 1$), so you cannot generate a Farey series in this way.

There is also an interesting connection with Pick's theorem, which concerns triangles on a lattice or grid. Suppose that the grid points are at $(m, n)$, where $m$ and $n$ are non-negative integers. Then Pick's theorem says (rather mysteriously) that the area of the triangle with vertices $(0, 0)$, $(a, b)$ and $(c, d)$ is $\frac{1}{2}N_s + N_i - 1$, where $N_s$ is the number of grid points which lie on the sides of the triangle and $N_i$ is the number of grid points inside the triangle. This generalises to any polyhedron on the lattice. Note in connection with the above problem that the area of the triangle is also $\frac{1}{2}|bc - ad|$.

**Solution to Question 20**

(i) Since we are dealing with integers, $a < b$ is the same as $a \leqslant b - 1$. Therefore, $bc \geqslant (a + 1)c$, as $c > 0$.

Similarly, $d < c$ implies that $ad \leqslant a(c - 1)$, as $a > 0$, i.e. $-ad \geqslant -a(c - 1)$.

Putting these together gives

$$bc - ad \geqslant (a + 1)c - a(c - 1) \;\text{ i.e. }\; bc - ad \geqslant c + a$$

as required.

(ii) We have $a < pb$ and $pd < c$, from the given inequality. Obviously, $pb$ and $pd$ are integers, so part (i) of this question applies, with $b$ replaced by $pb$ and $d$ replaced by $pd$. Therefore

$$(pb)c - a(pd) \geqslant a + c,$$

which is the required result.

(iii) This time we use the first part with $a$ replaced by $qa$, $b$ by $pb$, $c$ by $qc$ and $d$ by $pd$. We have $qa < pb$ and $pd < qc$. These are all integers, so

$$(pb)(qc) - (qa)(pd) \geqslant (qa) + (qc), \qquad (*)$$

which gives the required result when $q$ is cancelled (remember that $q$ is positive, and so non-zero) and the equation is divided by $(bc - ad)$, which is also positive, since $a/b < c/d$.

The inequality given in the question can also be written

$$\frac{d}{c} < \frac{q}{p} < \frac{b}{a},$$

so equation $(*)$ above is true with $p$ and $q$, $a$ and $d$, and $b$ and $c$ interchanged in pairs. This results in the given inequality for $q$.

For the very last part, take $a = 8$, $b = 9$, $c = 9$ and $d = 10$. Then any fraction $p/q$ lying between these given fractions must, by part (iii), satisfy

$$p \geqslant \frac{17}{1}, \qquad q \geqslant \frac{19}{1}.$$

Since we need $q$ less than 20, and certainly $p < q$ since $p/q < 9/10$, the only possibilities are 17/19 and 18/19. But we must be careful: the conditions derived above are only necessary and not sufficient. Therefore we must check that these fractions do indeed lie between 8/9 and 9/10. It is easy to see (not with a calculator, *please*!) that 18/19 is too big and $8/9 < 17/19 < 9/10$. Thus the only possibility is 17/19.

**Question 21**

Show by means of a sketch that if $0 \leqslant \mathrm{f}(t) \leqslant \mathrm{g}(t)$ for $0 \leqslant t \leqslant x$ then

$$0 \leqslant \int_0^x \mathrm{f}(t)\,\mathrm{d}t \leqslant \int_0^x \mathrm{g}(t)\,\mathrm{d}t. \tag{1}$$

Starting from the inequality $0 \leqslant \cos t \leqslant 1$, prove that if $0 \leqslant x \leqslant \frac{1}{2}\pi$ then $0 \leqslant \sin x \leqslant x$ and hence that $1 - \frac{1}{2}x^2 \leqslant \cos x \leqslant 1$. Deduce that

$$\frac{1}{1800} \leqslant \int_0^{\frac{1}{10}} \frac{x}{(2+\cos x)^2}\,\mathrm{d}x \leqslant \frac{1}{1797}.$$

By proving suitable inequalities for $\sin x$, show that

$$\frac{1}{3000} \leqslant \int_0^{\frac{1}{10}} \frac{x^2}{(1-x+\sin x)^2}\,\mathrm{d}x \leqslant \frac{2}{5999}.$$

**Discussion**

This is another of those questions which leads you through the first part and then asks you to apply the method yourself for the second part.

The first paragraph involves what might be thought of as a fairly obvious point, namely that bigger functions have bigger areas under their graphs. However, it is the sort of thing that has to be proved rigorously at higher levels (not here). A rigorous proof would have to start from a more sophisticated definition of integration. It may seem pointless to make something apparently easy into something difficult, but the point is that it is necessary to find a definition which works in more general situations — in higher dimensions, for example, or with some of the weird functions which turn out to be useful in mathematics. (The Dirac delta function springs to mind: $\delta(x)$ is defined by the conditions that it is zero except at $x = 0$ and the area under it is 1.)

You might notice that the inequalities on the trig. functions involve the first terms of their series. Can you write down inequalities of this form involving $x^n$?

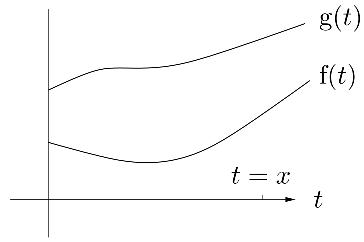
Just to check the result, I googled 'direct integral calculator' which offered me

`www.solvemymath.com/online_math_calculator/calculus/definite_integral/`

No doubt there are many other sites. This one approximates the first integral as 0.0005564825 (compare with 1/1797 = 0.0005564830) and the second integral as 0.000333388877 (compare this with 2/5999 = 0.000333388898) — not bad!

**Solution to Question 21**

The first integral in equation (1) in the question is equal to the area under the lower curve in the figure and the difference between the two integrals is equal to the area between the curves, both areas being obviously positive.




For the first inequality, we take $\mathrm{f}(t) = \cos t$ and $\mathrm{g}(t) = 1$ in equation (1) overleaf, giving

$$0 \leqslant \int_0^x \cos t \, \mathrm{d}t \leqslant \int_0^x 1 \, \mathrm{d}t \quad \text{i.e.} \quad 0 \leqslant \sin x \leqslant x. \tag{2}$$

Next we take $\mathrm{f}(t) = \sin t$ and $\mathrm{g}(t) = t$, noting that $0 \leqslant \sin t \leqslant t$ by the previous inequality (2). Thus

$$0 \leqslant \int_0^x \sin t \, \mathrm{d}t \leqslant \int_0^x t \, \mathrm{d}t \quad \text{i.e.} \quad 0 \leqslant -\cos x + 1 \leqslant \tfrac{1}{2}x^2,$$

which we can rewrite to obtain the required result:

$$1 - \tfrac{1}{2}x^2 \leqslant \cos x \leqslant 1\,. \tag{3}$$

This last inequality implies

$$\frac{1}{3 - \frac{1}{2}x^2} \geqslant \frac{1}{2 + \cos x} \geqslant \frac{1}{3}$$

from which (using a small extension of the basic result (1) we obtain

$$\int_0^{\frac{1}{10}} \frac{x}{(3 - \frac{1}{2}x^2)^2} \, \mathrm{d}x \geqslant \int_0^{\frac{1}{10}} \frac{x}{(2 + \cos x)^2} \, \mathrm{d}x \geqslant \int_0^{\frac{1}{10}} \frac{x}{3^2} \, \mathrm{d}x.$$

i.e.

$$\left. \frac{1}{3 - \frac{1}{2}x^2} \right|_0^{\frac{1}{10}} \geqslant \int_0^{\frac{1}{10}} \frac{x}{(2 + \cos x)^2} \, \mathrm{d}x \geqslant \left. \frac{x^2}{18} \right|_0^{\frac{1}{10}}.$$

This gives the required result when the limits are put in.

For the last part, note that the inequality (3) can be integrated (again using a small extension of the basic result) to give $x - \frac{1}{6}x^3 \leqslant \sin x \leqslant x$, from which we obtain

$$\frac{1}{1 - \frac{1}{6}x^3} \geqslant \frac{1}{1 - x + \sin x} \geqslant 1.$$

Therefore,

$$\int_0^{\frac{1}{10}} \frac{x^2}{(1 - \frac{1}{6}x^3)^2} \, \mathrm{d}x \geqslant \int_0^{\frac{1}{10}} \frac{x^2}{(1 - x + \sin x)^2} \, \mathrm{d}x \geqslant \int_0^{\frac{1}{10}} x^2 \, \mathrm{d}x$$

and

$$\left. \frac{2}{1 - \frac{1}{6}x^3} \right|_0^{\frac{1}{10}} \geqslant \int_0^{\frac{1}{10}} \frac{x^2}{(1 - x + \sin x)^2} \, \mathrm{d}x \geqslant \left. \frac{x^3}{3} \right|_0^{\frac{1}{10}}.$$

Evaluating the limits leads to the required result.

**Question 22**

The function f satisfies the condition $f'(x) > 0$ for $a \leqslant x \leqslant b$, and g is the inverse of f. By making a suitable change of variable, prove that

$$\int_a^b f(x)\,dx = b\beta - a\alpha - \int_\alpha^\beta g(y)\,dy, \tag{1}$$

where $\alpha = f(a)$ and $\beta = f(b)$. Interpret this formula geometrically, by means of a sketch, in the case where $\alpha$ and $a$ are both positive. Verify the result (1) for $f(x) = e^{2x}$, $a = 0$, $b = 1$.

Prove similarly and interpret the formula

$$2\pi \int_a^b x f(x)\,dx = \pi(b^2\beta - a^2\alpha) - \pi \int_\alpha^\beta \big[g(y)\big]^2\,dy. \tag{2}$$

**Discussion**

As is often the case, the required change of variable can be guessed by inspection of the limits.

To find the inverse function (NB: inverse, not reciprocal) to the function f, it is often best to try to think of the function g such that $g\big(f(x)\big) = x$, though making $x$ the subject of $y = f(x)$ is perhaps safer with an unfamiliar function.

The condition $f'(x) > 0$ ensures that f has a unique inverse; a function such as sin which has maximum and minimum points has a unique inverse only on restricted ranges of its argument which do not contain the turning points. (This is obvious from a sketch). The condition $f'(x) < 0$ would do equally well.

The geometrical interpretations of these formulae are exceptionally pleasing, though the second one needs some artistic skill to make it convincing.

**Solution to Question 22**

The limits of the integral on the right hand side of (1) are $\mathrm{f}(a)$ and $\mathrm{f}(b)$, which suggests the change of variable $y = \mathrm{f}(x)$. Making this change, so that $\mathrm{d}y = \mathrm{f}'(x)\mathrm{d}x$, gives

$$\int_\alpha^\beta \mathrm{g}(y)\,\mathrm{d}y = \int_a^b \mathrm{g}\big(\mathrm{f}(x)\big)\,\mathrm{f}'(x)\,\mathrm{d}x = \int_a^b x\,\mathrm{f}'(x)\,\mathrm{d}x.$$

For the last equality, we have used the definition of g as the inverse of f, i.e. $\mathrm{g}\big(\mathrm{f}(x)\big) = x$. This last integral is begging to be integrated by parts:

$$\int_a^b x\,\mathrm{f}'(x)\,\mathrm{d}x = x\mathrm{f}(x)\Big|_a^b - \int_a^b \mathrm{f}(x)\,\mathrm{d}x,$$

which gives the required result after evaluating $x\mathrm{f}(x)$ at $a$ and $b$. The first sketch below shows these areas: the area between the large and small rectangles is $(b\beta - a\alpha)$, which is split into the areas represented by the two integrals of equation (†), hatched vertically and horizontally, respectively.

Setting $\mathrm{f}(x) = e^{2x}$ and $a = 0$, $b = 1$ in the left hand side of (1) gives $\displaystyle\int_0^1 e^{2x}\,\mathrm{d}x = \tfrac{1}{2}\big(e^2 - 1\big)$. For the right hand side of (1), we have $\alpha = 1$ and $\beta = e^2$ so $b\beta - a\alpha = e^2$. The inverse of $e^{2x}$ is $\frac{1}{2}\ln y$, because $\frac{1}{2}\ln e^{2x} = \frac{1}{2}(2x) = x$, as required. The integral becomes

$$\int_1^{e^2} \tfrac{1}{2}\ln y\,\mathrm{d}y = \tfrac{1}{2}(y\ln y - y)\Big|_1^{e^2} = \tfrac{1}{2}(2e^2 - e^2) - \tfrac{1}{2}(0-1) = \tfrac{1}{2}e^2 + \tfrac{1}{2}.$$

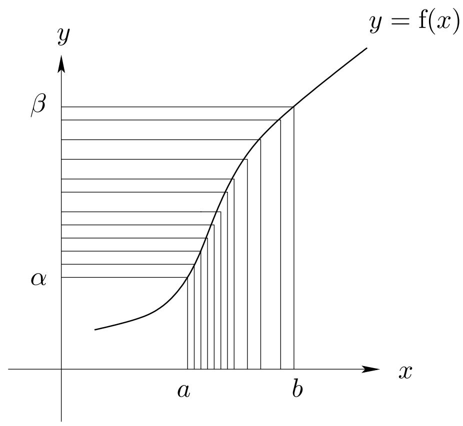
Thus the left hand side of (1) agrees with the right hand side.

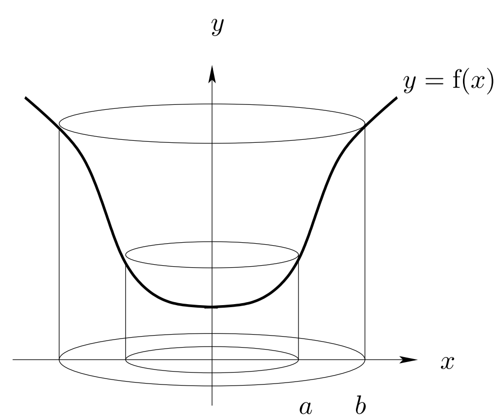
For the second formula, we can use the same method (change of variable followed by integration by parts):

$$\int_\alpha^\beta \big[\mathrm{g}(y)\big]^2\,\mathrm{d}y = \int_a^b x^2\mathrm{f}'(x)\,\mathrm{d}x = x^2\mathrm{f}(x)\Big|_a^b - \int_a^b 2x\mathrm{f}(x)\,\mathrm{d}x = (b^2\beta - a^2\alpha) - 2\int_a^b x\mathrm{f}(x)\,\mathrm{d}x,$$

which gives the required formula (2) on multiplication by $\pi$.

The first of the integrals in (2) is the volume of the solid body under the surface formed by rotating the curve $y = \mathrm{f}(x)$ round the $y$-axis; this volume is thought of as a set of concentric cylindrical shells of height $\mathrm{f}(x)$, internal radius $x$ and thickness $\mathrm{d}x$. The second integral is the volume inside the surface formed by rotating the curve $y = \mathrm{f}(x)$ round the $y$-axis; this volume is thought of as a pile of infinitesimally thin discs of radius $\mathrm{g}(y)$ and thickness $\mathrm{d}y$. The geometrical interpretation is therefore as shown in the second sketch below.



**Question 23**

A tennis tournament is arranged for $2^n$ players. It is organised as a knockout tournament, so that only the winners in any given round proceed to the next round. Opponents in each round except the final are drawn at random, and in any match either player has a probability $\frac{1}{2}$ of winning. Two players are chosen at random before the start of the first round. Find the probabilities that they play each other:

(i) in the first round;

(ii) in the final round;

(iii) in the tournament.

**Discussion**

Note that the set-up is not the usual one for a tennis tournament, where the only random element is in the first round line-up. Two players cannot then meet in the final if they are in the same half of the draw.

Part (i) is straightforward, but parts (ii) and (iii) need a bit of thought. There is an easy way and a hard way of tackling the problem, and both have merits. Close your eyes now if you do not want to receive the following hint for the easy way. Hint: think of the total number of different matches (i.e. pairings of the players), each of which is equally probable.

It is a good plan to check your answers, if possible, by reference to simple special cases where you can see what the answers should be; $n = 1$ or $n = 2$, for example.

Interestingly, the answers are independent of the probability that the players have of winning a match; the 2's in the answers represent the number of players in each match rather than (the reciprocal of) the probability that each player has of winning a match. It also does not matter how the draw for each round is made. This is clear if you use the 'easy' method mentioned above.

**Solution to Question 23**

Call the two players $P_1$ and $P_2$.

(i) Once $P_1$ has been given a slot, there are $2^n - 1$ slots for $P_2$, in only one of which will he or she play $P_1$. The probability of $P_1$ playing $P_2$ is therefore

$$\frac{1}{2^n - 1}.$$

(Note that this works for $n = 1$ and $n = 2$.)

(ii) **Long way**. To meet in the final, $P_1$ and $P_2$ must each win every round before the final, and must also not meet before the final. The probability that $P_1$ and $P_2$ do not meet in the first round and that they both win their first round matches, is

$$\left(1 - \frac{1}{2^n - 1}\right)\frac{1}{4} = \frac{1}{2}\left(\frac{2^{n-1} - 1}{2^n - 1}\right).$$

The probability that they win each round and do not meet before the final (i.e. for $n-1$ rounds) is

$$\frac{1}{2}\left(\frac{2^{n-1} - 1}{2^n - 1}\right) \times \frac{1}{2}\left(\frac{2^{n-2} - 1}{2^{n-1} - 1}\right) \times \cdots \times \frac{1}{2}\left(\frac{2^1 - 1}{2^2 - 1}\right) = \frac{1}{2^{n-1}}\frac{1}{2^n - 1}.$$

(ii) **Short way**. Since all processes are random here, the probability that any one pair contests the final is the same as that for any other pair. There are a total of $2^n(2^n - 1)/2$ different pairs, so the probability for any given pair is $1/[2^n(2^n - 1)/2]$.

(iii) **Long way**. We need to add the probabilities that $P_1$ and $P_2$ meet in each round. The probability that they meet in the $k$th round is the probability that they reach the $k$th round times the probability that they meet in the $k$th round given that they reach it, the latter (conditional) probability being $1/(2^{n-k+1} - 1)$, as can be inferred from part (i). As in part (ii), the probability that they reach the $k$th round is

$$\frac{1}{2^{k-1}}\frac{2^{n-k+1} - 1}{2^n - 1},$$

so the probability that they meet in the $k$th round is

$$\frac{1}{2^{k-1}}\frac{2^{n-k+1} - 1}{2^n - 1} \times \frac{1}{2^{n-k+1} - 1} = \frac{1}{2^{k-1}}\frac{1}{2^n - 1}.$$

Summing this from $k = 1$ to $n$ gives $1/2^{n-1}$.

(iii) **Short way**. By the same short argument as in part (ii), the probability of a given pair meeting in any given match (not necessarily the final) is $1/[2^n(2^n - 1)/2]$. Since the total number of matches is

$$2^n - 1,$$

(because one match is needed to knock out each player, and all players except one get knocked out) the probability of a given pair playing is

$$\frac{2^n - 1}{2^n(2^n - 1)/2} = \frac{1}{2^{n-1}}.$$

**Question 24**

A particle of mass $m$ is attached to a light circular rigid hoop of radius $a$ which is free to roll in a vertical plane on a rough horizontal table. Initially the hoop stands with the particle at the highest point. It is then displaced slightly. Show that while the hoop is rolling on the table, the speed $v$ of the particle when the radius to the particle makes an angle $\theta$ with the upward vertical is given by

$$v = 2\,(ag)^{\frac{1}{2}} \sin \tfrac{1}{2}\theta.$$

Write down expressions in terms of the variable $\theta$ for the horizontal displacement, $x$, of the particle from its initial position, and its height, $y$, above the table. Hence, or otherwise, show that

$$\dot{\theta} = (g/a)^{\frac{1}{2}} \tan \tfrac{1}{2}\theta$$

and

$$\ddot{y} = -2g \sin^2 \tfrac{1}{2}\theta.$$

By considering the reaction of the table on the hoop, or otherwise, describe what happens to prevent the hoop rolling beyond the position for which $\theta = \pi/2$.

**Discussion**

This gem is given in Littlewood's brilliant book, A Mathematician's Miscellany (Cambridge University Press). Apparently one of his colleagues set it annually to engineering students.

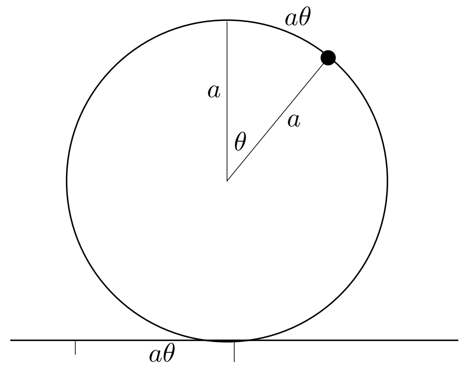
At first sight, it all seems fairly straightforward (though perhaps not simple): conservation of energy will give the speed of the particle at any height. You then need a bit of geometry to express $x$ and $y$ in terms of $a$ and $\theta$. The 'hence' before the second displayed formula means by differentiating; however, if you realise that the motion of the particle is an instantaneous rotation about the point of contact of the hoop and the plane, with angular velocity $\dot{\theta}$, you can instead use the standard formula $v = r\omega$ to find $\theta$. The reaction follows from use of Newton's law (force=mass $\times$ acceleration).

Nothing remarkable about this, I hear you say. BUT: suppose the particle reaches the bottom of the hoop ($\theta = \pi$). Then it has no potential energy; and it has no kinetic energy because it is instantaneously in contact with the table. What has happened to its energy?

The answer is that the motion cannot continue this far. In the ideal world of this problem, the hoop must 'levitate': it must break contact with the table and rise above it. In the real world, the mass of the hoop enters into the problem; furthermore, if the plane is not perfectly rough, slipping will reduce or destroy the effect. However, the effect will persist if the hoop has a small mass and high coefficient of friction with the table. I tried it with a coin taped to a sausage-shaped balloon, which is not exactly ideal, but it seemed to work.

The curve traced out by a point on the circumference of the hoop, given by $x = a(1 + \sin\theta)$, $y = a\cos\theta$, is called a cycloid. The motion described in the problem is equivalent to releasing a particle from the top of a smooth cycloid. A similar calculation shows that the particle will take off before reaching the lowest point, which corresponds to the hoop leaving the table.

**Solution to Question 24**




By conservation of energy (KE+PE), we have

$$0 + mg(2a) = \tfrac{1}{2}mv^2 + mg(a + a\cos\theta), \qquad \text{i.e.} \qquad v^2 = 2ag(1-\cos\theta) = 4ag\sin^2\tfrac{1}{2}\theta,$$

using the standard double-angle formula, as required.

The motion of the particle in Cartesian coordinates is given by

$$x = a\theta + a\sin\theta, \qquad y = a + a\cos\theta$$

so

$$\dot{x} = a(1+\cos\theta)\dot{\theta} \qquad \dot{y} = -a\sin\theta\,\dot{\theta}. \qquad (*)$$

Thus

$$v^2 = \dot{x}^2 + \dot{y}^2 = a^2\dot{\theta}^2(1+\cos\theta)^2 + a^2\dot{\theta}^2\sin^2\theta = a^2\dot{\theta}^2(2+2\cos\theta) = 4a^2\dot{\theta}^2\cos^2\tfrac{1}{2}\theta.$$

We have used $\cos^2\theta + \sin^2\theta = 1$ in the third equality and the standard double-angle formula in the fourth. Comparing this with the previous formula for $v$ gives the required formula for $\dot{\theta}$. Note that since the particle is instantaneously rotating about the bottom of the hoop, a distance $2a\cos(\tfrac{1}{2}\theta)$ away, with angular velocity $\dot{\theta}$, the formula for the speed of a rotating point, $v =$ radius $\times$ angular velocity, gives the above formula for $\dot{\theta}$ immediately.

To find $\ddot{y}$, we substitute the new expression for $\dot{\theta}$ into $(*)$ and differentiate, substituting for $\dot{\theta}$ yet again:

$$\begin{aligned} \dot{y} &= -a\sin\theta\,(g/a)^{\frac{1}{2}}\tan\tfrac{1}{2}\theta = -2(ga)^{\frac{1}{2}}\sin^2\tfrac{1}{2}\theta \\ \Longrightarrow \quad \ddot{y} &= -2(ga)^{\frac{1}{2}}\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta\,\dot{\theta} = -2(ga)^{\frac{1}{2}}\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta \times (g/a)^{\frac{1}{2}}\tan\tfrac{1}{2}\theta, \end{aligned}$$

which simplifies to the given answer.

To find the reaction, note that the vertical reaction $R$ of the table on the hoop must equal the vertical reaction of the particle on the hoop, since there are no other vertical forces on the hoop, and this is equal to the vertical reaction of the hoop on the particle. Equating mass times acceleration to vertical forces (upwards) on the particle therefore gives

$$m(-2g\sin^2\tfrac{1}{2}\theta) = -mg + R$$

so

$$R = mg(1 - 2\sin^2\tfrac{1}{2}\theta) = mg\cos\theta.$$

This vanishes when when $\theta = \pi/2$, at which time the hoop loses contact with the table.

**Question 25**

The random variable $B$ is normally distributed with mean zero and unit variance. Find the probability that the quadratic equation

$$X^2 + 2BX + 1 = 0$$

has real roots.

Given that the two roots $X_1$ and $X_2$ are real, find, giving your answers to three significant figures:

(i) the probability that both $X_1$ and $X_2$ are greater than $\frac{1}{5}$;

(ii) the expected value of $|X_1 + X_2|$.

**Discussion**

It is quite difficult to find statistics questions at this level that are not too difficult and are also not simple applications of standard methods. For example, $\chi^2$ tests are not really suitable, because the theory is sophisticated while the applications are usually rather straightforward. Most questions in the probability/statistics area tend therefore to concentrate on probability, and many of these have a bit of pure mathematics thrown in.

Here, the random variable is the coefficient of a quadratic equation, which is rather pleasing. But you have to handle the inequalities carefully.

If you don't have tables handy, just leave the answers in algebraic form; the numbers are not very interesting (but it is worth checking that, except for the answer to part (ii), the answers are less than 1).

**Solution to Question 25**

The solution of the quadratic is

$$X = -B \pm \sqrt{B^2 - 1}$$

which has real roots if $|B| \geqslant 1$. Let $\Phi(z)$ be the probability that a standard normally distributed variable $Z$ satisfies $Z \leqslant z$. Then the required answer is

$$\Phi(-1) + \big(1 - \Phi(1)\big) = 2 - 2\,\Phi(1) = 0.3174.$$

(i) We need the smaller root to be greater than $\frac{1}{5}$. The smaller root is $-B - \sqrt{B^2 - 1}$. Now provided $\sqrt{B^2 - 1}$ is real, we have

$$\begin{aligned}
-B - \sqrt{B^2 - 1} > \tfrac{1}{5} \quad &\Leftrightarrow \quad B + \tfrac{1}{5} < -\sqrt{B^2 - 1} \\
&\Leftrightarrow \quad (B + \tfrac{1}{5})^2 > B^2 - 1 \qquad \text{and} \quad (B + \tfrac{1}{5}) < 0 \\
&\Leftrightarrow \quad \tfrac{2}{5}B + \tfrac{1}{25} > -1 \qquad \text{and} \quad B < -\tfrac{1}{5} \\
&\Leftrightarrow \quad -\tfrac{1}{5} > B > -\tfrac{13}{5}.
\end{aligned}$$

However, if $B < -\frac{1}{5}$, then the condition that $\sqrt{B^2 - 1}$ is real, i.e. $|B| > 1$, implies the stronger condition $B < -1$. The condition that both roots are real *and* greater than $\frac{1}{5}$ is therefore

$$-\tfrac{13}{5} < B < -1$$

and the probability that both roots are real and greater than $\frac{1}{5}$ is

$$\Phi(-1) - \Phi(-2.6) = \Phi(2.6) - \Phi(1) = 0.9953 - 0.8413 = 0.1540.$$

The conditional probability that both roots are greater than $\frac{1}{5}$ *given* that they are real is

$$\begin{aligned}
\mathrm{P}\big(\text{both roots} > \tfrac{1}{5} \;\big|\; \text{both real}\big) \quad &= \quad \frac{\mathrm{P}\big(\text{ both roots} > \tfrac{1}{5} \textit{ and } \text{both roots real}\big)}{\mathrm{P}\big(\text{ both roots real}\big)} \\
&= \quad \frac{0.1540}{0.3174} = 0.485.
\end{aligned}$$

(ii) The sum of the roots is $-2B$, so we want the expectation of $|2B|$ given that $|B| > 1$, which is

$$\frac{\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{-1} (-2x)e^{-\frac{1}{2}x^2}\,\mathrm{d}x + \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{1}^{\infty} 2xe^{-\frac{1}{2}x^2}\,\mathrm{d}x}{\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{-1} e^{-\frac{1}{2}x^2}\,\mathrm{d}x + \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{1}^{\infty} e^{-\frac{1}{2}x^2}\,\mathrm{d}x} = \frac{2 \times \dfrac{1}{\sqrt{2\pi}} \times 2e^{-\frac{1}{2}}}{2\big(1 - \Phi(1)\big)} = 3.05.$$

### Question 26

A uniform straight thin rod of mass $m$ and length $4a$ lies on a smooth horizontal table. It is free to rotate about its midpoint. Initially the rod is at rest. A particle of mass $m$ sliding on the table with speed $u$ at right angles to the rod collides with the rod at a distance $a$ from its centre. The collision is perfectly elastic. Show that the angular speed, $\omega$, of the rod after the collision satisfies

$$a\omega = 6u/7.$$

Show also that the particle and rod collide again.

### Discussion

You will need to know (or to be able to calculate — which is not too hard) the moment of inertia of a rod. You will also have to use standard formulae for rotating rigid bodies, so this is testing the extremes of the A-level Further Mathematics mechanics course.

There are two approaches to this problem. The most straightforward conceptually is to conserve angular momentum about the pivot and kinetic energy. Angular momentum is conserved because the only external force on the system is in the form of a reaction impulse on the rod at the pivot and this does not affect the angular momentum about the pivot. Energy is conserved because the pivot is fixed so the impulse does no work. Alternatively, you could try Newton's law for elastic collisions (coefficient of restitution = 1) instead of conservation of energy, an elastic collision being one for which energy is conserved. (No energy is lost in the form of heat or sound: you cannot hear an elastic collision.)

The subsequent collision is interesting. Note that you are not asked to find exactly when it occurs (which would be more tricky) or to show that there is only one more collision (even harder). For those interested in the problem, a little variation would be to find the value of $k$ such that if the initial collision is a distance $ka$ along the rod, there is a subsequent collision.

**Solution to Question 26**

Conservation of angular momentum and energy give (respectively)

$$mua = mva + I\omega,$$

$$\tfrac{1}{2}mu^2 = \tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2,$$

where $I$ is the moment of inertia of the rod and $v$ is the velocity of the particle after impact (which is parallel to its original velocity since the impact is perpendicular).

Setting $I = 4ma^2/3$, we can rewrite these equations as

$$u - v = 4a\omega/3 \tag{1}$$

$$u^2 - v^2 = 4a^2\omega^2/3. \tag{2}$$

dividing the equation (2) by equation (1) gives

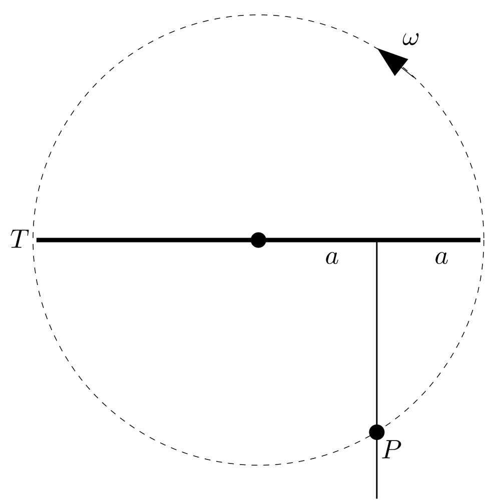
$$u + v = a\omega. \tag{3}$$

Note that equation (3) is exactly Newton's law for elastic collision (coefficient of restitution = 1):

$$a\omega - v = (-1)(0 - u).$$

Solving equations (1) and (3) simultaneously gives

$$a\omega = 6u/7, \qquad v = -a\omega/6.$$




The disc in the above diagram represents all points passed through by points on the rod. If the tip $T$ of the rod reaches the point labelled $P$ before the particle reaches it, then there will be a subsequent collision.

The angle turned through by the rod to reach this point is $\pi/2 + \sin^{-1}(1/2) = 2\pi/3$, which takes time $2\pi/3\omega$.

The distance that the particle has to travel is $\sqrt{3}a$, which takes time $\sqrt{3}a/v$. Substituting the expression found above for $v$ in terms of $a\omega$ shows that there must be another collision, since

$$\frac{2\pi}{3} < 6\sqrt{3}.$$

**Question 27**

Definite integrals can be evaluated approximately using the *trapezium rule*:

$$\int_{x_0}^{x_N} \mathrm{f}(x)\,\mathrm{d}x \approx \tfrac{1}{2}h\left[\mathrm{f}(x_0)+2\mathrm{f}(x_1)+\cdots+2\mathrm{f}(x_{N-1})+\mathrm{f}(x_N)\right], \tag{1}$$

where the interval length $h$ is determined by $Nh = x_N - x_0$, and $x_r = x_0 + rh$. Give a sketch to explain the basis of this approximation.

Use the trapezium rule with intervals of unit length to evaluate approximately the integral

$$\int_1^n \ln x\,\mathrm{d}x,$$

where $n$ is an integer (and $n > 2$). Deduce that

$$n! \approx e\,n^{\frac{1}{2}}\left(\frac{n}{e}\right)^n,$$

and determine, by means of a sketch, whether this approximate value exceeds $n!$.

Show using the trapezium rule that

$$(2n+1)!! \approx e^{-n}(2n+1)^{n+1},$$

where the double factorial notation indicates the product of odd integers $1.3.5\ldots(2n+1)$.

**Discussion**

This is just a matter of comparing the value of the trapezium approximation with the exact result. (Some rearranging is required to get the answers in the given form.)

The last part needs an obvious modification to the interval length. Note that you can get a better approximation to $(2n+1)!!$ by writing it as $(2n+1)!/(2^n n!)$ (the denominator is $2.4.6\ldots 2n$) and using the approximation (1) twice.

The approximations obtained here are related to the famous Stirling approximation

$$n! \approx \sqrt{2\pi n}(n/e)^n.$$

Stirling gave a much stronger result. He produced a series of which this is the first term (actually, his first term differs slightly from the familiar one above, which is due to de Moivre). His series has the curious property that although it fails to converge (the sum of the first $m$ terms tends to $\infty$ as $m \to \infty$), it can be truncated after a suitable number of terms to give an extraordinarily accurate approximation, provided $n$ is large enough. Stirling's derivation of the approximation required amazing ingenuity, at a time (1715) when very little calculus was known and all texts were written in cumbersome Latin. His identification of the constant factor, which arises in the form $\ln\sqrt{2\pi}$ in his proof, is remarkable. Note that this constant is not given accurately by the elementary approximation of this question. In fact, it can only be extracted by very sophisticated means, nowadays usually involving the identity $\int_{-\infty}^{\infty}\exp(-x^2)\,\mathrm{d}x = \sqrt{\pi}$.

**Solution to Question 27**

Setting $\mathrm{f}(x) = \ln x$, $x_0 = 1$, $x_N = n$, $h = 1$ and $N = n - 1$ in formula (1) overleaf gives

$$\int_1^n \ln(x)\,\mathrm{d}x \quad \approx \quad \tfrac{1}{2}\big(\ln 1 + 2\ln 2 + \cdots + 2\ln(n-1) + \ln n\big),$$

i.e.

$$\Big[x\ln x - x\Big]_1^n \quad \approx \quad \big(\ln 1 + \ln 2 + \cdots + \ln n\big) - \tfrac{1}{2}\ln 1 - \tfrac{1}{2}\ln n$$

i.e.

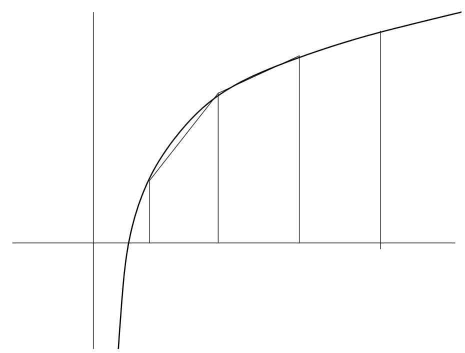
$$n\ln n - n + 1 \quad \approx \quad \ln n! - \tfrac{1}{2}\ln n.$$

Writing this as $(n+\frac{1}{2})\ln n - n + 1 \approx \ln n!$ and exponentiating gives $n^{n+\frac{1}{2}}e^{-n+1} \approx n!$, which is the required result.

From the sketch of the graph of $\ln x$ (which is not accurate: certain features have been emphasised), we see that the trapeziums lies below the graph (this is what is meant by saying that ln is a *concave* function), so the area of the trapeziums is less than the area under the curve. To get from the areas to the approximation we exponentiated, which will preserve the inequality:

$$a < b \Rightarrow \exp a < \exp b.$$

Now $\log n!$ occurs as the main component of the area under trapeziums and so is less than $n\log n - n + 1 + \frac{1}{2}\log n$. Thus the approximation exceeds $n!$.



For the second part, we repeat the above calculation using interval length 2, replacing $n$ by $(2n+1)$:

$$\int_1^{2n+1} \ln(x)\,\mathrm{d}x \quad \approx \quad \ln 1 + 2\ln 3 + \cdots + 2\ln(2n-1) + \ln(2n+1),$$

i.e.

$$\Big[x\ln x - x\Big]_1^{2n+1} \quad \approx \quad 2\Big(\ln 1 + \ln 3 + \cdots + \ln(2n+1)\Big) - \ln 1 - \ln(2n+1)$$

i.e.

$$(2n+2)\ln(2n+1) - 2n \quad \approx \quad 2\ln(2n+1)!!.$$

Exponentiating as before gives the required result.

**Question 28**

The set $S$ consists of $N$ (with $N > 2$) elements $a_1, a_2, \ldots, a_N$. It is acted upon by a binary operation $\circ$ defined by

$$a_j \circ a_k = a_m,$$

where $m$ is equal to the greater of $j$ and $k$. Determine, giving reasons, which of the four group axioms hold for $S$ under $\circ$ and which do not.

The set $S$ is acted upon by a binary operation $*$ defined by

$$a_j * a_k = a_n,$$

where $n = |j - k| + 1$. Determine, giving reasons, which of the group axioms hold for $S$ under $*$ and which do not.

**Discussion**

This problem involves a careful investigation of the group axioms. You should write them down accurately before you start. Sometimes, it is a good plan to look at the multiplication table; if any row or column is not a permutation of the elements of the set, then the set is not a group. (The property of the group table that every row and column contains every element exactly once is called the Latin Square property: sudoku is a special case.) If the size $N$ of the set were fixed at, say, $N = 4$, this would be a good method of proof, but it does not work as well for arbitrary size $N$. Nevertheless, it will help you to understand the multiplication. Note that both $\circ$ and $*$ are commutative ($a \circ b = b \circ a$), so you do not have to worry about left and right inverses, etc.

For many group theory problems, the associativity property is either trivial (a group of numbers, for example) or assumed (a group of matrices, for example — it is usually far to cumbersome to verify that matrix multiplication is associative). Here, associativity plays an important role.

Although some Further Mathematics A-level syllabuses do contain group theory, it is not a feature of the STEP syllabuses, and so never occurs in STEP questions.

**Solution to Question 28**

The four group axioms are:

(i) closure (product of any two elements is in the set);

(ii) existence of identity (unit) element (there exists an element $e$ such that $ea = ae = a$ for all $a$);

(iii) existence of inverses (for each $a$, there exists an element $b$ such $ab = ba = e$);

(iv) associativity ($(ab)c = a(bc)$ for all elements $a$, $b$ and $c$).

For $\circ$:

(i) $S$ is closed since $a_m$ is either $a_j$ or $a_k$.

(ii) The identity is $a_1$ since $a_j \circ a_1 = a_1 \circ a_j = a_j$ for any $a_j$.

(iii) To find the inverse to $a_j$, we need to find $a_k$ such that $a_j \circ a_k = a_1$. This is only possible if $a_j = a_1$, so this axiom does not hold.

(iv) We have

$$(a_j \circ a_k) \circ a_l = a_{\max(j,k)} \circ a_l = a_{\max(\max(j,k),l)},$$

where $\max(i,j)$ means 'the greater of $i$ and $j$'. Now

$$\max\big(\max(j,k), l\big) = \max(j,k,l) = \max\big(j, \max(k,l)\big)$$

so $(a_j \circ a_k) \circ a_l = a_j \circ (a_k \circ a_l)$ and $\circ$ is associative.

For $*$:

(i) $S$ is closed, since $1 \leqslant |j-k|+1 \leqslant N$.

(ii) The identity is $a_1$, since $|j-1|+1 = j-1+1 = j$.

(iii) To find the inverse to $a_j$, we need to find $a_k$ such that $|j-k|+1 = 1$. Clearly, the inverse of $a_j$ is $a_j$.

(iv) $*$ is associative if and only if $(a_j * a_k) * a_l = a_j * (a_k * a_l)$, i.e.

$$\big|(|j-k|+1) - l\big| + 1 = \big|j - (|k-l|+1)\big| + 1$$

for all positive integers $j$, $k$ and $l$. Should we try to prove this or should we seek a counterexample?

We could start constructing a group table in a simple case, say $N = 3$. We would then soon see a problem: the second row of the table is $a_2$, $a_1$, $a_2$ which is not a permutation of $a_1$, $a_2$, $a_3$. (Each row and column of the group table should be an arrangement of the $N$ elements of the group.) Since the first three group axioms hold, the problem must lie with the associativity.

To find a counterexample, we look at simple cases first. Let us set $j = k$ and see what happens. For associativity, we would then need (dropping the $+1$'s)

$$\big|1 - l\big| = \big|j - (|j-l|+1)\big|.$$

This holds if $l \leqslant j$, but if $l > j$, the right hand side is $|2j - l - 1|$ and this is certainly not equal to $l - 1$ in general. Therefore, $*$ is not associative.

The connection between associativity and the Latin Square property of the group table mentioned above is not obvious. Suppose that group axioms (i), (ii) and (iii) hold, but that two entries in a given row of the group table, say the row corresponding to the element $a$, are the same element $c$. Then there are two elements $b_1$ and $b_2$ which satisfy $a * b_1 = c$ and $a * b_2 = c$. But $a * b = c \Rightarrow a^{-1} * (a * b) = a^{-1} * c$ and if $*$ is associative this reduces to $b = a^{-1} * c$, giving a unique solution for $b$ after all. The multiplication cannot therefore be associative, and {axioms (i), (ii), (iii) but not Latin Square} implies {not axiom (iv)}.

**Question 29**

A damped system with feedback is modelled by the equation

$$\mathrm{f}'(t) = -\mathrm{f}(t) + k\mathrm{f}(t-1), \tag{1}$$

where $k$ is a given non-zero constant. Show that (non-zero) solutions for f of the form $\mathrm{f}(t) = Ae^{mt}$, where $A$ and $m$ are constants, are possible provided $m$ satisfies

$$m + 1 = ke^{-m}. \tag{2}$$

Show also, by means of a sketch or otherwise, that equation (2) can have 0, 1 or 2 real roots, depending on the value of $k$, and find the set of values of $k$ for which such solutions exist. For what set of values of $k$ do the corresponding solutions of (1) tend to zero as $t \to \infty$?

**Discussion**

Do not be put off by the words at the very beginning of the question, which you do not need to understand. However, their meaning can be gleaned from the equation (sometimes called a *differential-difference* equation). The left hand side of the equation is the rate of increase of the function f, which may measure the amplitude of some physical disturbance. According to the equation, this is equal to the sum of two terms. One is $-\mathrm{f}(t)$, which represents exponential damping: this term alone would give exponentially decreasing solutions. The other is a positive term proportional to $\mathrm{f}(t-1)$: this is called a feedback term, because it depends on the value of f one year (say) previously. The feedback term could represent some seasonal effect, while the damping term may be caused by some resistance-to-growth factor.

The suggested method of solving the equation is similar to that for second order linear equations with constant coefficients: you guess a solution ($e^{mt}$) and then substitute into the equation to check that this is a possible form of solution and to find the values of $m$ which will work. It is always possible to multiply the exponential by a constant factor, since any constant multiple of a solution is also a solution. (This is a consequence of the *linearity* of the equation: i.e. no terms involving $\mathrm{f}(t)^2$, $\mathrm{f}(t)^3$, etc.) We can also take linear combinations of solutions to produce a more general solution.

To find the set of values of $k$ which give real solutions of (2), you need to investigate the borderline case, where the two curves $y = m + 1$ and $y = ke^{-m}$ just touch (and therefore have the same gradient). Remember that the sign of $k$ is not restricted.

Unlike the second order differential equation case, the equation for $m$ is not quadratic; in fact, it is not even polynomial, since it has an exponential term. This means that there may not be exactly 2 solutions; there may be many solutions (e.g. the non-polynomial equation $\sin m = 0$ has solutions $m = 0, \pi, 2\pi, \ldots$) or even no solutions (e.g. $e^m = 0$ has no solutions). Therefore, we have no idea whether the method is going to work, or, if it does, whether there are other solutions of a different form which ought to be included in the general solution. The full analysis of equations such as (1) provides an extraordinarily rich field of study with many surprising results.

**Solution to Question 29**

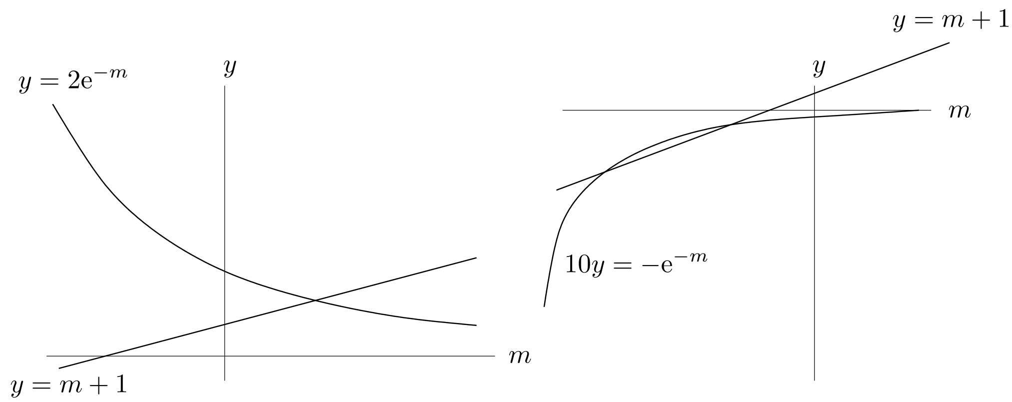
As suggested, we substitute $Ae^{mt}$ into equation (1), and cancel the overall factor $A$:

$$me^{mt} = -e^{mt} + ke^{m(t-1)}.$$

We can cancel the overall factor $e^{mt}$, as for a second order differential equation, but an exponential remains on account of the $\mathrm{f}(t-1)$ term:

$$m = -1 + ke^{-m},$$

which is equivalent to equation (2) overleaf.




The above sketches show $y = m + 1$ and $y = ke^{-m}$ on the same axes, for positive and negative $k$: in the first sketch, $k = 2$; in the second $k = -0.1$.

As can be seen from the sketches, equation (2) always has exactly one solution if $k > 0$. For $k < 0$, there may be two or zero solutions depending on whether the line and curve intersect, or just one solution if they touch. They will touch if there is a value of $m$ such that

$$m + 1 = ke^{-m} \quad \text{and} \quad 1 = -ke^{-m},$$

the second of these equations being the condition that the gradients (differentiate with respect to $m$ to find the gradient) of the line and curve are the same at the common point. Solving the two equations gives $m = -2$ and $k = -e^{-2}$.

If $0 > k > -e^{-2}$, then the curve and line will intersect, so the set of values of $k$ for which equation (2) has solutions is $k > 0$ and $-e^{-2} \leqslant k < 0$.

The corresponding solutions to equation (1) tend to zero as $t \to \infty$ if and only if $m < 0$, because then they are exponentially decreasing rather than increasing.

If $k < 0$, we can see from the sketch that the solutions of (2), if there are any (i.e. if $-e^{-2} \leqslant k < 0$), occur in the bottom left quadrant, and so $m < 0$. The corresponding solutions of (1) will tend to zero.

If $k > 0$, the intersection of the graph of $y = ke^{-m}$ with the $y$-axis is at $y = k$ whereas the intersection of the graph of $y = m + 1$ with the $y$-axis is at $y = 1$. For $k \geqslant 1$, the solution of (2) occurs in the top right quadrant and has $m \geqslant 0$. For $0 < k < 1$, the solution occurs in the top left quadrant and has $m < 0$.

The range of $k$ for which the solutions tend to zero as $t \to \infty$ is therefore $-e^{-2} \leqslant k < 0$ and $0 < k < 1$.

**Question 30**

A train of length $l_1$ and a lorry of length $l_2$ are heading for a level crossing at speeds $u_1$ and $u_2$ respectively. Initially, the front of the train and the front of the lorry are at distances $d_1$ and $d_2$ from the crossing. Find the conditions on $u_1$ and $u_2$ under which a collision will occur. On a diagram with $u_1$ and $u_2$ measured along the $x$ and $y$ axes, respectively, shade the region which represents collision.

Hence show that, if $u_1$ and $u_2$ are two independent random variables both uniformly distributed on $(0, V)$, then the probability of a collision in the case when the back of the train is nearer to the crossing than the front of the lorry is

$$\frac{l_1 l_2 + l_2 d_1 + l_1 d_2}{2 d_2 (l_2 + d_2)}.$$

Find the probability of a collision in the other two possible cases.

**Discussion**

This is a bivariate problem involving uniform distributions, so the whole thing boils down to finding areas in the $x$-$y$ plane where the axes represent the random variables. Once you have realised this, there is not a lot of probability involved.

The three different cases which arise are obvious from a diagram. (In one case, all the shaded area lies above the diagonal.) Translated back to the set-up of the problem, the significance of the first case is that there will be no collision in this case if $u_1 > u_2$.

**Solution to Question 30**

The collision will occur if either the front of the train reaches the crossing while the lorry is on it, or the front of the lorry reaches the crossing while the train is on it, i.e.

$$\frac{d_2}{u_2} \leqslant \frac{d_1}{u_1} \leqslant \frac{d_2+l_2}{u_2} \quad \text{or} \quad \frac{d_1}{u_1} \leqslant \frac{d_2}{u_2} \leqslant \frac{d_1+l_1}{u_1}.$$

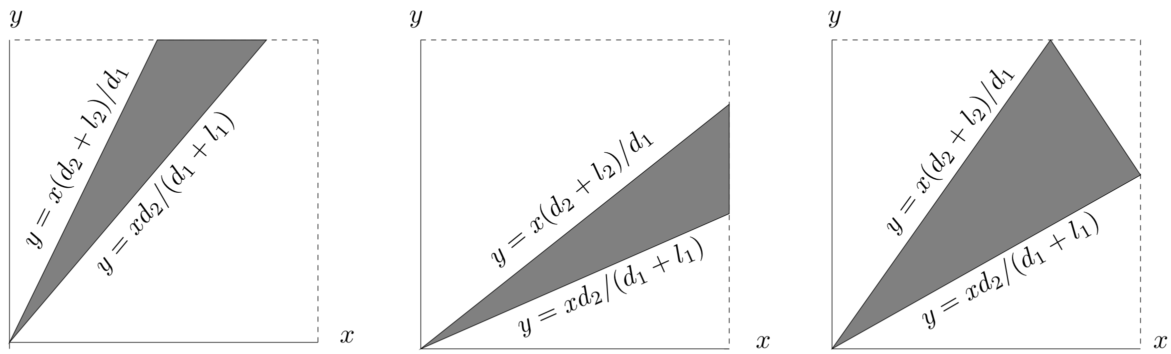
Making $u_2$ the 'subject' of these inequalities gives

$$\frac{d_2}{d_1}u_1 \leqslant u_2 \leqslant \frac{d_2+l_2}{d_1}u_1 \quad \text{or} \quad \frac{d_2}{d_1+l_1}u_1 \leqslant u_2 \leqslant \frac{d_2}{d_1}u_1,$$

which can be combined:

$$\frac{d_2}{d_1+l_1}u_1 \leqslant u_2 \leqslant \frac{d_2+l_2}{d_1}u_1.$$

Note that $d_2/(d_1+l_1) \leqslant (d_2+l_2)/d_1$ (always). The inequalities are represented by the shaded areas shown below, shown in the three possible cases determined by the positions of the lines $y = x(d_2+l_2)/d_1$ and $y = xd_2/(d_1+l_1)$ relative to the diagonal $y = x$. (The distinction between the three pictures is not required at this stage.)




If both $u_1$ and $u_2$ are uniformly distributed on $(0, V)$, then the probability of the speeds lying in any region of the $x$-$y$ plane (within the square of side $V$) is equal to the area of the region divided by $V^2$. All we have to do is calculate the areas of the shaded regions in the three diagrams.

The first diagram has $d_2/(d_1+l_1) > 1$, which means that the back of the train is closer to the crossing than the front of the lorry. The area of the shaded region is half height of the triangle $\times$ base, i.e.

$$\tfrac{1}{2} \times V \times \left(\frac{d_1+l_1}{d_2}V - \frac{d_1}{l_2+d_2}V\right),$$

which simplifies to the given answer.

The second diagram has $(d_2+l_2)/d_1 < 1$, which means that the back of the lorry is closer to the crossing than the front of the train. The probability is just obtained from the previous case by interchanging $1 \leftrightarrow 2$ in the previous answer.

The third diagram represents the intermediate case. The shaded area above the diagonal is

$$\tfrac{1}{2}\left(V - \frac{d_1}{l_2+d_2}V\right) \times V$$

and the area below is the same with $1 \leftrightarrow 2$, so the combined probability is

$$1 - \frac{d_1}{2(l_2+d_2)} - \frac{d_2}{2(l_1+d_1)}.$$

**Question 31**

Two identical snowploughs plough the same stretch of road. The first starts at a time $t_1$ seconds after it starts snowing, and the second starts from the same point $t_2 - t_1$ seconds later, going in the same direction. Snow falls so that the depth of snow increases at a constant rate of $k$ ms$^{-1}$. The speed of each snowplough is $ak/z$ ms$^{-1}$ where $z$ is the depth (in metres) of the snow it is ploughing and $a$ is a constant. Each snowplough clears all the snow. Show that the time $t$ at which the second snowplough has travelled a distance $x$ metres satisfies the equation

$$a\frac{\mathrm{d}t}{\mathrm{d}x} = t - t_1 e^{x/a}. \tag{\dagger}$$

Hence show that the snowploughs will collide when they have travelled $a(t_2/t_1 - 1)$ metres.

**Discussion**

There is something exceptionally beautiful about this question, but it is hard to identify exactly what it is; seeing the question for the first time makes even hardened mathematicians smile with pleasure.

There is a modelling element to it: you have to set up equations from the information given in the text. The first equation you need is a simple first order differential equation to find the time taken by the first snowplough to travel a distance $x$. The corresponding equation for the second snowplough is a bit more complicated, because the depth of snow at any point depends on the time at which the first snowplough reached that point, clearing the snow.

The differential equation (†) can be solved using an integrating factor. However, the equation which arises naturally at this point is one involving $\mathrm{d}x/\mathrm{d}t$, which cannot (apparently) be solved by any means. It is the rather good trick of turning the equation upside down (regarding $t$ as a function of $x$ instead of $x$ as a function of $t$) that allows the problem to be solved so neatly.

There is a generalisation to $n$ identical snowploughs . . . .

**Solution to Question 31**

Suppose that the the first snowplough reaches a distance $x$ at time $T$ after it starts snowing. Then the depth of snow it encounters is $kT$ and its speed is therefore $ak/(kT)$, i.e. $a/T$. The equation of motion of the first snowplough is

$$\frac{\mathrm{d}x}{\mathrm{d}T} = \frac{a}{T}.$$

Integrating both sides with respect to $T$ gives

$$x = a \ln T + \text{constant of integration.}$$

We know that $T = t_1$ when $x = 0$ (the snowplough started $t_1$ seconds after the snow started), so

$$x = a \ln T - a \ln t_1.$$

This can be rewritten as

$$T = t_1 e^{x/a}.$$

When the second snowplough reaches $x$ at time $t$, snow has been falling for a time $t - T$ since it was cleared by the first snowplough, so the depth at time $t$ is $k(t - T)$ metres, i.e. $k(t - t_1 e^{x/a})$ metres. Thus the equation of motion of the second snowplough is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{ak}{k(t - t_1 e^{x/a})}.$$

Now we use the standard result

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 1 \Big/ \frac{\mathrm{d}t}{\mathrm{d}x}$$

to obtain the required equation (†).

Multiplying by $e^{-x/a}$ (an *integrating factor*) and rearranging gives

$$e^{-x/a}\frac{\mathrm{d}t}{\mathrm{d}x} - \frac{e^{-x/a}t}{a} = -\frac{t_1}{a} \quad \text{i.e.} \quad \frac{\mathrm{d}}{\mathrm{d}x}\left(te^{-x/a}\right) = -\frac{t_1}{a}$$

which integrates to

$$te^{-x/a} = -\frac{t_1}{a}x + \text{constant of integration.}$$

Since the second snowplough started ($x = 0$) at time $t_2$, the constant of integration is just $t_2$ and the solution is

$$t = (t_2 - t_1 x/a)e^{x/a}.$$

The snowploughs collide when they reach the same position at the same time. Let this position be $x = X$. Then

$$T = t \Longrightarrow t_1 e^{X/a} = (t_2 - t_1 X/a)e^{X/a},$$

so $X$ is given by

$$t_1 = (t_2 - t_1 X/a).$$

This is equivalent to the required formula.

**Question 32**

Give a careful argument to show that, if $G_1$ and $G_2$ are distinct subgroups of a finite group $G$ such that every element of $G$ is in at least one of $G_1$ or $G_2$, then either $G_1 = G$ or $G_2 = G$.

Give an example of a group $H$ which has subgroups $H_1$, $H_2$ and $H_3$ such that every element of $H$ is in at least one of $H_1$, $H_2$ or $H_3$ and $H_1 \neq H$, $H_2 \neq H$, $H_3 \neq H$.

**Discussion**

This is fairly typical of the sort of problem one encounters in abstract mathematics at higher levels. At first sight, it seems completely smooth: there is no obvious handhold to get you started. The best way to start is to try to understand why the result might be true, but this can require a lot of experience. Alternatively, you could write down something that looks like a plausible beginning and then look at it and hope for inspiration. For example, you could start with 'Let $g_1 \in G_1$ and $g_2 \in G_2$'. It is often useful to draw a diagram (e.g. a Venn diagram showing the groups).

Probably, you should aim to construct a proof by contradiction: assume that neither $G_1$ nor $G_2$ is equal to $G$ and look for a contradiction.

For the second part, you have to demonstrate your understanding of the first part by producing a counterexample in the three subgroup case. The best strategy, if you can't see immediately what to do, is to try a few examples and see why they are not counterexamples (with a bit of luck, they may be). The golden rule is to start with the very simplest example first.

Another way of writing the result proved in the first part is:

if $G_1$ and $G_2$ are finite groups, then $G_1 \cup G_2$ is a group if and only if $G_1 \subseteq G_2$ or $G_2 \subseteq G_1$.

**Solution to Question 32**

First we assume that the result we are trying to prove does not hold, i.e. that

$$\text{neither } G_1 \text{ nor } G_2 \text{ is equal to } G. \qquad (*)$$

Since $G_2 \neq G$, we can find an element $g_1 \in G$ such that $g_1 \notin G_2$. According to the question, every element is in at least one of $G_1$ or $G_2$, so $g_1 \in G_1$. Similarly, we can find $g_2 \in G_2$ with $g_2 \notin G_1$.

Where is $g_1 g_2$? It is certainly in $G$, by the group closure axiom, so according to the question it must be in at least one of $G_1$ or $G_2$.

Suppose first $g_1 g_2 \in G_1$. By the group inverse axiom, $g_1^{-1} \in G_1$, so $g_1^{-1}(g_1 g_2) \in G_1$ by the closure axiom, i.e. (using associativity) $g_2 \in G_1$. This is a contradiction.

Similarly, $g_1 g_2 \in G_2$ leads to a contradiction.

Therefore the hypothesis $(*)$ must be false; i.e. either $G_1 = G$ or $G_2 = G$.

For the case of three distinct subgroups, we need $H$ to have at least three distinct elements. Let us first consider the case $|H| = 3$:

$$H = \{e, h_1, h_2\}; \qquad H_1 = \{e\}, \quad H_2 = \{e, h_1\} \quad H_3 = \{e, h_2\},$$

where $e$ is the unit element. (Recall that every subgroup must contain the unit element.) For this to work, we must first have $h_1^2 = e = h_2^2$ (so that each subgroup is closed), which is not a problem. However, $h_1 h_2$ is a problem since it cannot equal $e$ (because $h_1 \neq h_2$), it cannot equal $h_1$ (because $h_2 \neq e$), nor can it equal $h_2$ (because $h_1 \neq e$). Therefore $H$ fails to satisfy group closure and there is no group of order 3 with three distinct subgroups.

Instead, let us consider

$$H = \{e,\, h_1,\, h_2,\, h_1 h_2\}; \qquad H_1 = \{e,\, h_1\}, \quad H_2 = \{e,\, h_2\} \quad H_3 = \{e,\, h_1 h_2\}.$$

As before, we need $h_1^2 = e = h_2^2$. Now if we take $h_1 h_2 = h_2 h_1$ we have a consistent group and consistent subgroups, and the conditions of the problem are fulfilled. It is the group of reflections of a rectangle, where $h_1$ and $h_2$ are reflections and $h_1 h_2$ is a rotation through $180°$.

### Question 33

A target consists of a disc of unit radius and centre $O$. A certain marksman never misses the target, and the probability of any given shot hitting the target within a distance $t$ from $O$ is $t^2$, where $0 \leqslant t \leqslant 1$. The marksman fires $n$ shots independently. The random variable Y is the radius of the smallest circle, with centre $O$, which encloses all the shots. Show that the probability density function of Y is $2ny^{2n-1}$, and show that the expected area of the circle is $\pi n/(n+1)$.

The shot which is furthest from $O$ is rejected. Show that the expected area of the smallest circle, with centre $O$, which encloses the remaining $(n-1)$ shots is $\left(\dfrac{n-1}{n+1}\right)\pi$.

The $k$ shots which are furthest from $O$ are rejected. Show that the expected area of the smallest circle, with centre $O$, which encloses the remaining $(n-k)$ shots is $\left(\dfrac{n-k}{n+1}\right)\pi$.

### Discussion

This is really just about 'order statistics': the distribution of the $(n-k)$th largest object.

Although we are not told about the angular distribution around the centre, one possibility is that the shots are random, being equally likely to fall in any small area: this would give a probability of a shot hitting any disc as proportional to the area of the disc, as required by the question.

The last part is an add-on: it was not part of the original question, and increases the difficulty considerably. (The extra difficulty comes from having to evaluate an integral by integrating $k$ times by parts.) However, the question appeared to be heading in this direction, so it seemed a pity not to let it finish.

**Solution to Question 33**

One way of thinking about this is to find the probability that $(n-1)$ of the shots are within a distance $y$ of the centre, and the $n$th is between $y$ and $y+\mathrm{d}y$ of the centre (so that no smaller circle is possible). This gives

$$n \times y^{2n-2} \times 2y\,\mathrm{d}y$$

where the factor $n$ is included because any of the $n$ shots could be the outermost shot.

Alternatively, one can find the probability that all shots are within a distance $y$ from the centre, which is $y^{2n}$, and differentiate to get the probability that the largest is between $y$ and $y+\mathrm{d}y$.

The area of the circle is $\pi y^2$, so the expected area is

$$\int_0^1 \pi y^2 (2ny^{2n-1})\,\mathrm{d}y = \frac{n}{n+1}\,\pi.$$

If we reject the outermost shot, the probability that radius of the circle is between $z$ and $z+\mathrm{d}z$ is

$$n(n-1) \times z^{2n-4} \times (1-z^2) \times 2z\,\mathrm{d}z,$$

the factors being: the number of ways the outermost shot and the second outermost shot can be chosen; the probability that $(n-2)$ lie within a distance $z$ from the centre; the probability that the $n$th shot is further than $z$ from the centre; the probability that the $(n-1)$th shot is between distances $z$ and $z+\mathrm{d}z$ from the centre. At this stage, we should integrate this expression from 0 to 1 to check that it really is a probability density function.

Again, we could find the probability that $(n-1)$ shots are within a distance $z$ from the centre, which is $nz^{2n-2}(1-z^2)+z^{2n}$ (the second term arising because $(n-1)$ shots will certainly be within within a distance $z$ from the centre if all $n$ are), and differentiate.

The expected area is thus

$$\int_0^1 \pi z^2 n(n-1)z^{2n-4}(1-z^2)2z\,\mathrm{d}z$$

To evaluate this integral most easily, we make the change of variable $u=z^2$ giving

$$\int_0^1 \pi n(n-1)u^{n-1}(1-u)\,\mathrm{d}u = \pi n(n-1)\left[\frac{1}{n}-\frac{1}{n+1}\right]$$

which simplifies to the given answer.

For the last part, the probability density function is

$${}^nC_k\,(n-k)v^{2(n-k-1)}(1-v^2)^k 2v.$$

The first term is the number of ways of choosing the $k$ shots outside radius $v$, the factor $(n-k)$ is the number of ways of choosing the shot with the largest radius from the remaining $(n-k)$.

The expected area, using the new variable $u=v^2$, is

$$\int_0^1 \pi\ {}^nC_k(n-k)u^{(n-k)}(1-u)^k\,\mathrm{d}u$$

Integrating by parts $k$ times gives

$$\int_0^1 \pi\ {}^nC_k(n-k)u^n\frac{k!}{(n-k+1)(n-k+2)\cdots n}\,\mathrm{d}u = \int_0^1 \pi(n-k)u^n\,\mathrm{d}u,$$

the ‘bits’ all vanishing since they are evaluated at 0 or 1. The final integral is $(n-k)/(n+1)$.

**Question 34**

The integral $I$ is defined by

$$I = \int_1^2 \frac{(2-2x+x^2)^k}{x^{k+1}} \mathrm{d}x,$$

where $k$ is a constant. Show that

$$\begin{aligned} I &= \int_0^1 \frac{(1+x^2)^k}{(1+x)^{k+1}}\, \mathrm{d}x && (1) \\ &= \int_0^{\pi/4} \frac{1}{\left[\sqrt{2}\, \cos\theta \cos(\frac{1}{4}\pi - \theta)\right]^{k+1}}\, \mathrm{d}\theta && (2) \\ &= 2\int_0^{\pi/8} \frac{1}{\left[\sqrt{2}\, \cos\theta \cos(\frac{1}{4}\pi - \theta)\right]^{k+1}}\, \mathrm{d}\theta. && (3) \end{aligned}$$

Hence show that

$$I = 2\int_0^{\sqrt{2}-1} \frac{(1+x^2)^k}{(1+x)^{k+1}} \mathrm{d}x. \qquad (4)$$

Deduce that

$$\int_1^{\sqrt{2}} \left(\frac{2-2x^2+x^4}{x^2}\right)^k \frac{1}{x}\, \mathrm{d}x = \int_1^{\sqrt{2}} \left(\frac{2-2x+x^2}{x}\right)^k \frac{1}{x}\, \mathrm{d}x.$$

**Discussion**

Most of the steps of this question involve changes of variables which are signalled by the changes in the limits of the integrals. To obtain (4) will need a calculation to verify that $\sqrt{2}-1$ is in fact what you hope it is. (The presence of $(1+x^2)$ in the integrand should reassure you if you are on the right track.)

There are easier ways of obtaining the final result. The virtue of the method suggested (apart from providing a good exercise in manipulating integrals) is that you reach a form of the integral (that of equation (2)) which has an explicit symmetry: the integrand is symmetric about $\frac{1}{8}\pi$, as can be seen by replacing $\theta$ by $\frac{1}{4}\pi - \theta$. It is this symmetry which allows the transformation.

Note that the value of the index $k$ plays no part in the problem. In fact, the final result holds not just for powers but for any function F:

$$\int_1^{\sqrt{2}} \mathrm{F}\big(2x^{-2} - 2 + x^2\big)\frac{1}{x}\, \mathrm{d}x = \int_1^{\sqrt{2}} \mathrm{F}\big(2x^{-1} - 2 + x\big)\frac{1}{x}\, \mathrm{d}x.$$

**Solution to Question 34**

To obtain equation (1) overleaf we make the change of variable $x = 1 + y$, which will obviously get the required limits and the correct denominator:

$$I = \int_1^2 \frac{(1+(x-1)^2)^k}{x^{k+1}} \mathrm{d}x = \int_0^1 \frac{(1+y^2)^k}{(1+y)^{k+1}} \mathrm{d}y.$$

Now just rename $y$ as $x$ (it does not matter what this dummy variable of integration is called since it is eventually going to be evaluated at 0 and 1).

The next change of variable is slightly more difficult to spot. The presence of $\pi/4$ in the limits suggests a trig. substitution and the fact that $\pi/4$ has to correspond to $x = 1$ makes $x = \tan\theta$ a sensible try. Any lingering doubt is removed by the noting that the factor $(1+x^2)$ in the numerator will become a convenient $\sec^2\theta$. We therefore try $x = \tan\theta$:

$$I = \int_0^1 \frac{(1+x^2)^k}{(1+x)^{k+1}}\,\mathrm{d}x = \int_0^{\pi/4} \frac{(\sec^2\theta)^k}{(1+\tan\theta)^{k+1}}\sec^2\theta\,\mathrm{d}\theta = \int_0^{\pi/4} \frac{1}{\big[\cos\theta(\cos\theta+\sin\theta)\big]^{k+1}}\,\mathrm{d}\theta.$$

This is equivalent to the integrand in (2), which involves $\cos(\frac{1}{4}\pi - \theta)$, as can be seen from the standard trig. formula

$$\cos(\tfrac{1}{4}\pi - \theta) = \cos(\tfrac{1}{4}\pi)\cos\theta + \sin(\tfrac{1}{4}\pi)\sin\theta = \tfrac{1}{\sqrt{2}}\,\cos\theta + \tfrac{1}{\sqrt{2}}\,\sin\theta.$$

Notice the factor of 2 in front of the integral in (3). This means that the integral from 0 to $\frac{1}{8}\pi$ must have the same value as the integral from $\frac{1}{8}\pi$ to $\frac{1}{4}\pi$, so that (3) follows from (2) by splitting the range of integration into two equal parts. This we can check by means of the change of variable $\phi = \frac{1}{4}\pi - \theta$:

$$\int_0^{\pi/8} \frac{1}{\big[\sqrt{2}\,\cos\theta\cos(\frac{1}{4}\pi-\theta)\big]^{k+1}}\,\mathrm{d}\theta = \int_{\pi/4}^{\pi/8} \frac{1}{\big[\sqrt{2}\,\cos(\frac{1}{4}\pi-\phi)\cos\phi\big]^{k+1}}\,(-\mathrm{d}\phi)$$

as claimed. (We can replace $\phi$ by $\theta$, and exchange the upper and lower limits because of the extra minus sign.)

Now if we make the transformation $\theta = \tan^{-1}x$ (i.e. we reverse the previous steps) we will get back to the integrand of (1), but the upper limit of the integral will be $\tan(\frac{1}{8}\pi)$ instead of $\tan(\frac{1}{4}\pi)$. We must therefore confirm that $\tan(\frac{1}{8}\pi) = \sqrt{2}-1$. Let $\tan(\frac{1}{8}\pi) = t$. Then

$$\tan(\tfrac{1}{4}\pi) = \frac{2t}{(1-t^2)},$$

i.e. $t^2 + 2t - 1 = 0$. Solving this equation using the quadratic formula, and taking the positive root since we know that $\tan(\frac{1}{8}\pi)$ is positive (the negative root gives $\tan(\frac{1}{2}\pi + \frac{1}{8}\pi)$) shows that $t = \sqrt{2}-1$ as required.

For the final part, we set $x = y - 1$ in (4) and $x = y^2$ in the original integral for $I$.

### Question 35

The Bernoulli polynomials, $\mathrm{B}_n(x)$ (where $n = 0,\ 1,\ 2, \ldots$), are defined by $\mathrm{B}_0(x) = 1$ and, for $n \geqslant 1$,

$$\frac{\mathrm{dB}_n}{\mathrm{d}x} = n\mathrm{B}_{n-1}(x) \tag{1}$$

and

$$\int_0^1 \mathrm{B}_n(x)\mathrm{d}x = 0. \tag{2}$$

(i) Show that $\mathrm{B}_4(x) = x^2(x-1)^2 + c$, where $c$ is a constant (which you need not evaluate).

(ii) Show that, for $n \geqslant 2$, $\mathrm{B}_n(1) - \mathrm{B}_n(0) = 0$.

(iii) Show, by induction or otherwise, that

$$\mathrm{B}_n(x+1) - \mathrm{B}_n(x) = nx^{n-1} \qquad (n \geqslant 1). \tag{3}$$

(iv) Hence show that

$$n\sum_{m=0}^{k} m^{n-1} = \mathrm{B}_n(k+1) - \mathrm{B}_n(0),$$

and deduce that $\sum\limits_{m=0}^{1000} m^3 = (500500)^2$.

### Discussion

The Swiss family Bernoulli included no fewer than eight mathematicians who were counted amongst the leading scholars of their day. They made major contributions to all branches of mathematics, especially differential calculus. There was great rivalry between some members of the family; between brothers Jakob (1654–1705) and Johann (1667–1748), in particular. Johann once published an important result in the form of a Latin anagram, in order to retain the priority of discovery without giving the game away to his brother.

The anagram was: $24a$, $6b$, $6c$, $8d$, $33e$, $5f$, $2g$, $4h$, $33i$, $6l$, $21m$, $26n$, $16o$, $8p$, $5q$, $17r$, $16s$, $25t$, $32u$, $4x$, $3y$, $+$, $-$, $-$, $\pm$, $=$, 4, 2, 1, $'$. The notation means that his important result contained, for instance, the letter $a$ 24 times either in text or in equations. After waiting for a year for someone to solve it, Bernoulli weakened and published the solution himself. If you are trying to solve the anagram yourself, you might like to know that it is about the Riccati equation $y' = ay^2 + bx^n$, which can be solved (very cunningly, as it turns out) when $n$ is of the form $-4m/(2m \pm 1)$ for any positive integer $m$. (Newton also published some work in the form of anagrams, during his conflict with Leibniz).

The polynomials described above were discovered by Jakob Bernoulli. They are defined recursively; that is to say, the zeroth polynomial is given an explicit value, and the $n$th is determined from the $(n-1)$th. Here, $\mathrm{B}_{n-1}$ has to be integrated to obtain $\mathrm{B}_n$, which means that $\mathrm{B}_n$ is a polynomial of degree $n$, and the constant of integration is determined by the 'normalisation' condition (2), so $\mathrm{B}_n$ is uniquely determined. We have to do this explicitly for part (i).

**Solution to Question 35**

(i) First we find $\mathrm{B}_1(x)$ by integrating $1 \times \mathrm{B}_0(x)$, using equation (1): $\mathrm{B}_1(x) = x + k$, where $k$ is a constant. We find $k$ by applying the condition $\int_0^1 \mathrm{B}_1(x)\mathrm{d}x = 0$, which gives $k = -\frac{1}{2}$. Next we find $\mathrm{B}_2(x)$ by similar means, giving $x^2 - x + 1/6$, and hence $\mathrm{B}_3(x) = x^3 - 3x^2/2 + x/2$ and $\mathrm{B}_4(x)$ as given.

(ii) Note that the question involves $\mathrm{B}_n(x)$ is evaluated at $x = 1$ and $x = 0$, i.e. at the limits of the integral (2). We therefore try the effect of integrating (1) between these limits:

$$\mathrm{B}_n(1) - \mathrm{B}_n(0) \equiv \int_0^1 \frac{\mathrm{dB}(x)}{\mathrm{d}x}\,\mathrm{d}x = n\int_0^1 \mathrm{B}_{n-1}(x)\,\mathrm{d}x = 0,$$

using property (2) with $n$ replaced by $(n-1)$.

(iii) First the easy bit of the induction proof. For $n = 1$, using our result from part (i),

$$\mathrm{B}_n(x+1) - \mathrm{B}_n(x) = (x + 1 - \tfrac{1}{2}) - (x - \tfrac{1}{2}) = 1 \equiv nx^{n-1},$$

so the formula holds.

Now suppose that it holds for $n = k$:

$$\mathrm{B}_k(x+1) - \mathrm{B}_k(x) - kx^{k-1} = 0. \tag{4}$$

and investigate

$$\mathrm{B}_{k+1}(x+1) - \mathrm{B}_{k+1}(x) - (k+1)x^k, \tag{5}$$

which we hope will also equal zero.

The only helpful thing we know about Bernoulli polynomials involves the derivatives. Therefore, let us see what happens when we differentiate the expression (5):

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{B}_{k+1}(x+1) - \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{B}_{k+1}(x) - (k+1)kx^{k-1}.$$

Now using (1) gives

$$(k+1)B_k(x+1) - (k+1)\mathrm{B}_k(x) - (k+1)kx^{k-1}.$$

Note that we have used the chain rule to differentiate $\mathrm{B}_{k+1}(x+1)$ with respect to $x$ rather than with respect to $(x+1)$. Note also that there is a pleasing overall factor of $(k+1)$, which suggests that we are on the right track. In fact, taking out this factor gives exactly the left hand side of equation (4), which is zero.

Of course, we are not finished yet: we have only shown that the derivative of equation (5) is equal to zero; the expression (5) is therefore constant:

$$\mathrm{B}_{k+1}(x+1) - \mathrm{B}_{k+1}(x) - (k+1)x^k = A.$$

We must show that $A = 0$. Setting $x = 0$ gives $\mathrm{B}_{k+1}(1) - \mathrm{B}_{k+1}(0) = A$, which implies that $A = 0$ by part (ii).

(iv) Summing (3) from $x = 0$ to $x = k$ gives the first of these results immediately because nearly all the terms cancel in pairs. The evaluation of the sum follows by calculating $\mathrm{B}_4(1001) - \mathrm{B}_4(0)$ from the result given in part (i).

**Question 36**

(i) The equation $x^2 + bx + c = 0$ has the property that if $k$ is a root, then $k^{-1}$ is a root. Show that either $c = 1$, or $c = -1$ and $b = 0$.

(ii) Instead of the above property, the equation $x^2 + bx + c = 0$ has the property that if $m$ is a root, then $1 - m$ is a root. Determine carefully the restrictions that this property places on $b$ and $c$.

(iii) The equation $x^2 + bx + c = 0$ has both the properties described in (i) and (ii) above. Show that $b = -1$ and $c = 1$.

(iv) The equation $x^3 + px^2 + qx + r = 0$ has both the properties described in (i) and (ii) above. Find the possible values of $p$, $q$ and $r$.

**Discussion**

This question was difficult to word, because of the possibility that $k^{-1}$ and $k$ need not be distinct. (For example, one should not write '$k^{-1}$ is *also* a root' in line 1, since this might seem to imply that $k$ is different from $k^{-1}$.) Note that the roots are not restricted to being real.

You will need to marshal your thoughts carefully for all parts of the problem. In part (i) you have to make sure that you have thought of all the possibilities, remembering that $k$ could be, for example, $+1$, in which case the given property relates the root to itself rather than to the other root. It is not a bad idea to start with 'Let the roots be $\alpha$ and $\beta$' so as not to keep referring to 'the other root'.

No extra work is required for part (iii): you only have to compare the results of parts (i) and (ii). Part (iv) needs careful organisation. One plan is to consider separately the case when the equation has repeated roots. Having dealt with this (using the previous part), you can concentrate on the equations with three different roots.

You may have noticed that the transformations which preserve the set of roots in parts (iii) and (iv) (i.e. $z \to z^{-1}$ and $z \to 1 - z$) form a group, the group multiplication being composition of the transformations (perform one transformation, then the next). Call these two transformations f and h respectively. Then we have the structure $\mathrm{f}^2 = \mathrm{h}^2 = 1$ and $\mathrm{f\,hf} = \mathrm{hf\,h}$ (verify this by choosing $z = 3$, say, or prove it for general $z$), so the distinct elements are 1, f, h, f h, hf and f hf. The group has order 6 and is isomorphic to the group of permutations of three objects and to the symmetry group (rotations and reflections) of a triangle. You can think of the group acting on points in the Argand diagram, where it will generally map a set of six points into itself; for example, if we start with the point $z = 3$, we get the set consisting of 3, $\mathrm{f}(3) = \frac{1}{3}$, $\mathrm{h}(3) = -2$, $\mathrm{fh}(3) = -\frac{1}{2}$, $\mathrm{hf}(3) = \frac{2}{3}$, $\mathrm{fhf}(3) = \frac{3}{2}$ (check this!). In just one special case, the group maps a set of two points into itself, namely $\frac{1}{2}(1 + i\sqrt{3})$ and $\frac{1}{2}(1 - i\sqrt{3})$. In just one special case, the group maps a set of three points into itself, namely 2, $\frac{1}{2}$ and $-1$ (in fact there is another possibility if we include $\infty$, namely 0, 1 and $\infty$). In these cases, the group can map the set of roots of certain quadratic or cubic equations into itself but in other cases there would have to be 6 roots for this to happen; the equation would have to be sextic at least.

**Solution to Question 36**

(i) Let the roots be $\alpha$ and $\beta$. If $\alpha = k$, then the condition in the question says that $k^{-1}$ is also a root. Therefore, either $\alpha = k^{-1}$, i.e. $k = \pm 1$, or $\beta = k^{-1}$. Thus the equation is either of the form

$$(x-k)(x-k^{-1}) = 0, \quad \text{i.e.} \quad x^2 + bx + 1 = 0 \tag{1}$$

with $b = -k - k^{-1}$, or of the form

$$(x-1)(x+1) = 0, \quad \text{i.e.} \quad x^2 - 1 = 0. \tag{2}$$

Note that equation (1) includes the possibilities $(x-1)(x-1) = 0$ and $(x+1)(x+1) = 0$. In equation (1), $b$ is arbitrary since any number $b$ (real or complex) can be written in the form $-k-k^{-1}$ for some $k$ (just solve the quadratic equation $b = -k - k^{-1}$ for $k$). The restrictions are therefore: $c = 1$ and $b$ is arbitrary; or $c = -1$ and $b = 0$.

(ii) This time, either the two roots sum to 1, or each root must satisfy $m = 1 - m$, i.e. $m = \frac{1}{2}$, which is in fact a special case of the roots summing to 1 case. The only equation is therefore

$$(x-m)(x-1+m) = 0, \quad \text{i.e.} \quad x^2 - x + c = 0, \tag{3}$$

with $c = m(1-m)$. Here, $c$ is unrestricted, since any number can be written in the form $m(1-m)$.

(iii) The only equation which is compatible with equation (3) and also with one of equations (1) or (2). is

$$x^2 - x + 1 = 0,$$

with roots $\frac{1}{2}(1 + i\sqrt{3})$ and $\frac{1}{2}(1 - i\sqrt{3})$.

(iv) Suppose first that the roots of the equation are all different and pick one which is not equal to $\frac{1}{2}$. Call this root $\alpha$. If $\alpha = m$ then, in order to satisfy property (ii), one of the other roots must equal $1-m$; call this root $\beta$ and the third root $\gamma$. We cannot have $\gamma = m$ or $\gamma = 1-m$ because then there would be a repeated root. In order to satisfy property (ii), we must therefore have $\gamma = \frac{1}{2}$, in which case either $\beta = 2$ or $\alpha = 2$ by property (i). The roots are therefore $\frac{1}{2}$, 2, and $-1$. Each of these satisfies both properties (i) and (ii), so one possible equation is

$$(x-2)(x+1)(x-\tfrac{1}{2}) = 0.$$

Suppose instead that there are at most two different roots. Then we can repeat the argument of part (iii), showing that the only possible roots are $\frac{1}{2}(1 + i\sqrt{3})$ and $\frac{1}{2}(1 - i\sqrt{3})$. This leads to the two equations

$$(x^2 - x + 1)\big(x - \tfrac{1}{2}(1 - i\sqrt{3})\big) = 0 \qquad \text{and} \qquad (x^2 - x + 1)\big(x - \tfrac{1}{2}(1 + i\sqrt{3})\big) = 0.$$

**Question 37**

A heavy particle lies on a smooth horizontal table and is attached to one end of a light inelastic string of length $L$. The other end of the string is attached to a point $A$ on the circumference of the base of a vertical post which is fixed onto the table. The base of the post is a circle of radius $a$ with its centre at a point $O$ on the table. Initially, at time $t = 0$, the string is taut and perpendicular to the radius $OA$. The particle is then struck in such a way that the string starts winding round the post and remains taut. At a later time $t$, length $a\,\theta$ $(< L)$ of the string is in contact with the post. Using cartesian axes, or otherwise, find the position and velocity vectors of the particle at time $t$ in terms of $a$, $L$, $\theta$ and $\dot{\theta}$, and hence, or otherwise, show that the speed of the particle is $(L - a\theta)\dot{\theta}$.

If the initial speed of the particle is $v$, show that the particle hits the post at time $L^2/(2av)$.

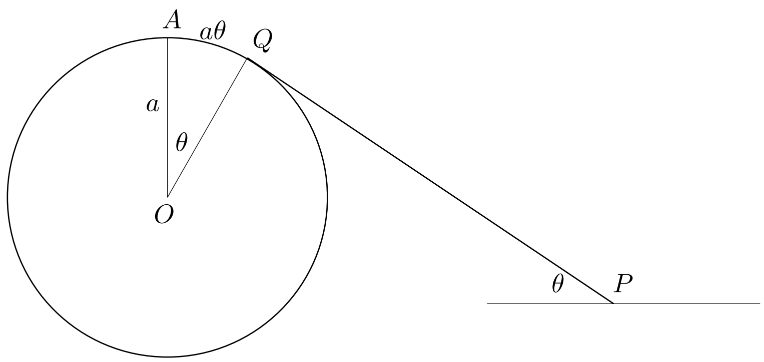
**Discussion**

This is a familiar situation: it occurs often in Tom and Jerry cartoons; or you may perhaps have tried it yourself with a conker on a string. You have probably noticed that the conker rotates faster and faster as the string shortens. In fact, at the very last moment, the angular speed is infinite. Of course, this can only happen in an idealised situation, but it is not uncommon in such situations for things to become infinite in a finite time. For example, an ideal ball bouncing on the spot bounces an infinite number of times before coming to rest in a finite time.

Do not be tempted to conserve angular momentum about the centre of the post. The line of the string does not pass through the centre of the post, so the tension in the string provides a couple about the centre which changes the angular momentum.

The last paragraph involves integration.

**Solution to Question 37**




The length of the arc round the post which is in contact with the string is $a\theta$, so $\angle AOQ = \theta$, where the point $Q$, marked in the diagram, is the last point of contact between the string and the post. At $Q$, the string is perpendicular to the radius of the post, since otherwise the string would be kinked. $Q$ has cartesian coordinates

$$(a\sin\theta, a\cos\theta)$$

with respect to the centre of the post. The length of the string in between $Q$ and the particle $P$ is $(L - a\theta)$. The cartesian coordinates of $P$ are therefore

$$\mathbf{r} = a(\sin\theta, \cos\theta) + (L - a\theta)(\cos\theta, -\sin\theta).$$

The velocity of the particle is given (by direct differentiation) by

$$\begin{aligned}
\dot{\mathbf{r}} &= a\dot{\theta}\,(\cos\theta, -\sin\theta) - a\dot{\theta}\,(\cos\theta, -\sin\theta) + (L - a\theta)\,(-\sin\theta, -\cos\theta)\dot{\theta} \\
&= (L - a\theta)(-\sin\theta, -\cos\theta)\,\dot{\theta}.
\end{aligned}$$

The speed of the particle (i.e. the magnitude of $\dot{\mathbf{r}}$) is therefore

$$(L - a\theta)\dot{\theta}.$$

This is in fact the expected result, because the particle is instantaneously rotating about $Q$ with angular speed $\dot{\theta}$. The 'otherwise' in the question is just to write down the answer immediately, but you would have to be very confident to do this.

Since energy is conserved, the speed of the particle is constant, and equal to the initial speed $v$. Thus we obtain the following first order differential equation for $\theta$:

$$v = (L - a\theta)\dot{\theta}.$$

Integrating both sides with respect to $t$ gives

$$vt = L\theta - \tfrac{1}{2}a\theta^2 + \text{constant}$$

and the constant is zero since $\theta = 0$ when $t = 0$. (If you are not happy with this integration, just differentiate the result to see how it works.)

The string is all wrapped round the post when $\theta = L/a$, which happens when $vt = L^2/a - \frac{1}{2}L^2/a$ as required.

Note that the final angular speed $\dot{\theta}$ is infinite when $L = a\theta$!

### Question 38

Describe fully the locus of the point in three-dimensional space whose position vector is $\mathbf{r}$ in each of the following cases:

(i) $(\mathbf{a}-\mathbf{b}).\mathbf{r} = \frac{1}{2}\big(a^2 - b^2\big)$;

(ii) $(\mathbf{a}-\mathbf{r}).(\mathbf{b}-\mathbf{r}) = 0$;

(iii) $|\mathbf{r}-\mathbf{a}|^2 = \frac{1}{2}|\mathbf{a}-\mathbf{b}|^2$;

(iv) $|\mathbf{r}-\mathbf{b}|^2 = \frac{1}{2}|\mathbf{b}-\mathbf{a}|^2$.

Here, $\mathbf{a}$ and $\mathbf{b}$ are fixed vectors with lengths $a$ and $b$.

Prove algebraically that equations (i) and (ii) are together equivalent to equations (iii) and (iv) together. Explain carefully the geometric meaning of this equivalence.

### Discussion

You have to be sure to give a full geometrical description of the loci. It is not enough to say 'a plane'; you need to explain which plane, by giving the normal and the distance from the origin, for example, or any three points on the plane.

To prove equivalence between sets of equations requires care. It is best to take it slowly, showing that one set of equations implies each of the equations of the other set, and vice versa.

The geometrical interpretations are the hardest part of this problem. You should draw pictures.

**Solution to Question 38**

(i) We have

$$(\mathbf{a}-\mathbf{b}).\mathbf{r} = \tfrac{1}{2}(a^2-b^2) = \tfrac{1}{2}(\mathbf{a}-\mathbf{b})(\mathbf{a}+\mathbf{b}) \ \text{ i.e. } \ (\mathbf{a}-\mathbf{b}).\big(\mathbf{r}-\tfrac{1}{2}(\mathbf{a}+\mathbf{b})\big) = 0,$$

which is the standard equation of a plane perpendicular to $(\mathbf{a}-\mathbf{b})$ through the point $\frac{1}{2}(\mathbf{a}+\mathbf{b})$. In other words, $P$ is the perpendicular bisector of the line from $\mathbf{a}$ to $\mathbf{b}$.

(ii) Expanding the dot product, remembering that $\mathbf{x}.\mathbf{x} = |\mathbf{x}|^2$ for any vector $\mathbf{x}$ and writing $r$ for $|\mathbf{r}|$, gives

$$r^2 - (\mathbf{a}+\mathbf{b}).\mathbf{r} + \mathbf{a}.\mathbf{b} = 0 \ \text{ i.e. } \ \big|\mathbf{r}-\tfrac{1}{2}(\mathbf{a}+\mathbf{b})\big|^2 = \tfrac{1}{4}\big|\mathbf{a}-\mathbf{b}\big|^2$$

which is the equation of a sphere, call it $S$, of radius $\frac{1}{2}|\mathbf{a}-\mathbf{b}|$ and centre $\frac{1}{2}(\mathbf{a}+\mathbf{b})$.

Alternatively, we can use the equation directly. Since $\mathbf{r}-\mathbf{a}$ is perpendicular to $\mathbf{r}-\mathbf{b}$, the point $\mathbf{r}$ lies on a sphere with diameter $\mathbf{b}-\mathbf{a}$, by the converse of the angle in a semi-circle theorem.

(iii) and (iv) The equations describe spheres, call them $S_1$ and $S_2$, each with radius $\frac{1}{\sqrt{2}}|\mathbf{a}-\mathbf{b}|$, and with centres $\mathbf{a}$ and $\mathbf{b}$ respectively.

First we prove that (i) and (ii) imply (iv) and also, by interchanging $\mathbf{a}$ and $\mathbf{b}$, (iii). Expanding (i) gives $\mathbf{a}.\mathbf{r} = \mathbf{b}.\mathbf{r} + \frac{1}{2}(a^2-b^2)$. Now we expand (ii) and use the above equation to eliminate $\mathbf{a}.\mathbf{r}$:

$$r^2 - \mathbf{a}.\mathbf{r} - \mathbf{b}.\mathbf{r} + \mathbf{a}.\mathbf{b} = 0 \Rightarrow r^2 - 2\mathbf{b}.\mathbf{r} - \tfrac{1}{2}(a^2-b^2) + \mathbf{a}.\mathbf{b} = 0.$$

Expanding (iv) gives exactly this last equation.

Next we prove that (iii) and (iv) imply (i). If we subtract (iv) from (iii) and expand, we get

$$(r^2 - 2\mathbf{a}.\mathbf{r} + a^2) - (r^2 - 2\mathbf{b}.\mathbf{r} + b^2) = 0 \ \text{ i.e. } \ 2(\mathbf{b}-\mathbf{a}).\mathbf{r} = b^2 - a^2,$$

which is equivalent to (i).

Finally, we prove that (iii) and (iv) imply (ii). This time, we add (iii) and (iv):

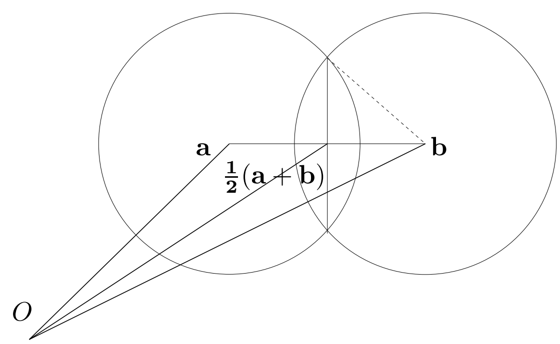
$$(r^2 - 2\mathbf{a}.\mathbf{r} + a^2) + (r^2 - 2\mathbf{b}.\mathbf{r} + b^2) = (a^2 + b^2 - 2\mathbf{a}.\mathbf{b}) \ \text{ i.e. } \ 2r^2 - 2\mathbf{a}.\mathbf{r} - 2\mathbf{b}.\mathbf{r} + 2\mathbf{a}.\mathbf{b} = 0$$

which is equivalent to (ii).

For the geometrical equivalence, we need to look at the set of points $\mathbf{r}$ that satisfy both (i) and (ii), and compare it with the set of points that satisfy both (iii) and (iv).

Let us do the easier case first. The intersection of $S_1$ and $S_2$ gives a circle, call it $C$, with centre $\frac{1}{2}(\mathbf{a}+\mathbf{b})$ whose plane is perpendicular to the line joining the centres, i.e. to $\mathbf{a}-\mathbf{b}$. The radius of $C$ is, by Pythagoras, (see diagram)

$$\sqrt{\tfrac{1}{2}|\mathbf{a}-\mathbf{b}|^2 - \tfrac{1}{4}|\mathbf{a}-\mathbf{b}|^2} = \tfrac{1}{2}|\mathbf{a}-\mathbf{b}|.$$




For the second case, note that the centre of $S$, namely $\mathbf{r} = \frac{1}{2}(\mathbf{a}+\mathbf{b})$, satisfies equation (i) and therefore lies in $P$. Thus $P$ cuts $S$ into hemispheres. The intersection is therefore a circle with centre $\mathbf{r} = \frac{1}{2}(\mathbf{a}+\mathbf{b})$ and radius $\frac{1}{2}|\mathbf{a}-\mathbf{b}|$ (i.e. the radius of $S$) whose plane is $P$. This is just the circle $C$ described above.

The geometrical meaning of the equivalence is therefore the two ways of describing the circle defined above: as the intersection of two spheres or as the intersection of a plane and a sphere.

**Question 39**

The matrix $\mathbf{F}$ is defined by

$$\mathbf{F} = \mathbf{I} + \sum_{n=1}^{\infty} \frac{1}{n!} t^n \mathbf{A}^n, \qquad (\dagger)$$

where $\mathbf{A} = \begin{pmatrix} -3 & -1 \\ 8 & 3 \end{pmatrix}$, and $t$ is a variable scalar.

(i) Evaluate $\mathbf{A}^2$, and show that

$$\mathbf{F} = \mathbf{I}\cosh t + \mathbf{A}\sinh t.$$

(ii) Verify that $\mathbf{F}^{-1} = \mathbf{I}\cosh t - \mathbf{A}\sinh t$, and show that $\dfrac{\mathrm{d}\mathbf{F}}{\mathrm{d}t} = \mathbf{FA}$.

(iii) The vector $\mathbf{r} = \begin{pmatrix} x \\ y \end{pmatrix}$ satisfies the differential equation

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} + \mathbf{Ar} = \mathbf{0},$$

with $x = a$ and $y = b$ at $t = 0$. Solve this equation by means of a suitable matrix integrating factor, and hence show that

$$\begin{aligned} x &= a\cosh t + (3a + b)\sinh t \\ y &= b\cosh t - (8a + 3b)\sinh t. \end{aligned}$$

**Discussion**

Please do not be put off by the unfamiliar ideas in this question. If you do not know about the hyperbolic functions $\cosh t$ and $\sinh t$, then look up the definitions and a few of their properties: you need their derivatives and their power series expansions (both closely related to those of the corresponding trigonometric functions).

You should find that $\mathbf{A}^2$ is exceptionally simple in form, which is why you can sum the series to find $\mathbf{F}$. The derivative of a matrix $\mathbf{F}$ is defined to be the matrix whose elements are the derivatives of the elements of $\mathbf{F}$ (i.e. do the obvious thing).

If you recall how integrating factors work, you will be able to do the last part by following your nose and using the copious hints from the first parts of the question.

The series in equation ($\dagger$), with $t = 1$, is just the exponential series and defines the exponential of the matrix $\mathbf{A}$. Integrating a single equation by means of an integrating factor requires use of exponentials and the same is true for matrix equations. You have to be careful with exponentials of matrices, because unlike ordinary numbers, matrices do not in general commute; this means that $\exp\mathbf{A}\exp\mathbf{B} \neq \exp(\mathbf{A} + \mathbf{B})$, in general.

Although $\mathbf{A}$ has been very carefully chosen here, the method above can be applied quite generally to systems of (any number of) linear differential equations with constant coefficients. The method fails if the coefficients in the differential equations are not constant (which is exactly the situation where the method of integrating factors works to greatest advantage for a single differential equation) because of the failure of the matrix to commute with its derivative.

**Solution to Question 39**

(i)

$$\mathbf{A}^2 = \begin{pmatrix} -3 & -1 \\ 8 & 3 \end{pmatrix} \begin{pmatrix} -3 & -1 \\ 8 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

All the matrices in the sum for $\mathbf{F}$ are either $\mathbf{A}$ (for odd $n$) or $\mathbf{I}$ (for even $n$). Collecting up these terms gives

$$\mathbf{F} = \Big(1 + \frac{1}{2!}t^2 + \frac{1}{4!}t^4 + \cdots\Big)\mathbf{I} + \Big(t + \frac{1}{3!}t^3 + \frac{1}{5!}t^5 + \cdots\Big)\mathbf{A}.$$

The coefficients of $\mathbf{I}$ and $\mathbf{A}$ in this expression are exactly the series for $\cosh t$ and $\sinh t$, respectively.

(ii) The verification of the formula for $\mathbf{F}^{-1}$ is straightforward:

$$\mathbf{F}^{-1}\mathbf{F} = (\mathbf{I}\cosh t - \mathbf{A}\sinh t)(\mathbf{I}\cosh t + \mathbf{A}\sinh t) = \mathbf{I}\cosh^2 t - \mathbf{A}^2\sinh^2 t = \mathbf{I}\cosh^2 t - \mathbf{I}\sinh^2 t = \mathbf{I}$$

since $\cosh^2 t - \sinh^2 t = 1$.

The derivative of $\cosh t$ is $\sinh t$ and vice-versa, so

$$\frac{\mathrm{d}\mathbf{F}}{\mathrm{d}t} = \mathbf{I}\sinh t + \mathbf{A}\cosh t = (\mathbf{I}\cosh t + \mathbf{A}\sinh t)\mathbf{A} = \mathbf{F}\mathbf{A},$$

as required.

(iii) Consider the equation

$$\frac{\mathrm{d}r}{\mathrm{d}t} + ar = 0,$$

where $r$ and $a$ are now ordinary functions, not matrices. On way of solving this is to multiply by a suitable integrating factor so that the equation becomes

$$\frac{\mathrm{d}(\mathrm{f}r)}{\mathrm{d}t} = 0,$$

which we can integrate trivially: $\mathrm{f}r = c$, where $c$ is a constant, so $r = \mathrm{f}^{-1}c$. Let us try the same idea on the given matrix differential equation. If we can write it as

$$\frac{\mathrm{d}(\mathbf{F}\mathbf{r})}{\mathrm{d}t} = \mathbf{0},$$

then differentiating the product gives (careful of the order of the matrices)

$$\mathbf{F}\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} + \frac{\mathrm{d}\mathbf{F}}{\mathrm{d}t}\mathbf{r} = \mathbf{0},$$

which is equivalent to the original equation multiplied on the left by $\mathbf{F}$, since $\mathbf{F}$ satisfies $\mathrm{d}\mathbf{F}/\mathrm{d}t = \mathbf{F}\mathbf{A}$. Therefore, $\mathbf{r} = \mathbf{F}^{-1}\mathbf{c}$, where $\mathbf{c}$ is a matrix with constant elements. To find $\mathbf{c}$, we set $t = 0$: we have $\mathbf{F}^{-1} = \mathbf{I}\cosh 0 - \mathbf{A}\sinh 0 = \mathbf{I}$, so at $t = 0$, $\mathbf{r} = \mathbf{c}$.

The solution to the given equation is therefore given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \big(\mathbf{I}\cosh t - \mathbf{A}\sinh t\big)\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \cosh t + 3\sinh t & +\sinh t \\ -8\sinh t & \cosh t - 3\sinh t \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix},$$

which gives the required solution when the matrices are multiplied out.

You may have spotted that we have solved not the original equation, but the original equation multiplied throughout by $\mathbf{F}$ which may not be equivalent. Here it *is* equivalent, because $\mathbf{F}$ is non-singular and can therefore be cancelled. (Its determinant is $+1$, as can be seen by direct calculation or by using the rather deep formula $\det(\exp\mathbf{A}) = \exp(\mathrm{trace}\mathbf{A})$, where trace $\mathbf{A}$ means the sum of the diagonal elements of $\mathbf{A}$, which is in this case 0.)

## Question 40

(i) The complex number $\omega$ is such that $\omega^2 - 2\omega$ is real. Sketch the locus of $\omega$ in the Argand diagram. If $\omega^2 = x + iy$, where $x$ and $y$ are real, describe fully and sketch the locus of points $(x, y)$ in the $x$-$y$ plane.

(ii) The complex number $t$ is such that $t^2 - 2t$ is imaginary. If $t^2 = p + iq$, where $p$ and $q$ are real, sketch the locus of points $(p, q)$ in the $p$-$q$ plane.

## Discussion

The most difficult part of this question is the sketch in the last part. *Please* do not use your graphics calculator. Instead, check what happens at large values of $p$, look for values of $p$ for which $q$ is not real, find where the curve meets the axes, look for symmetries, investigate gradients, and so on.

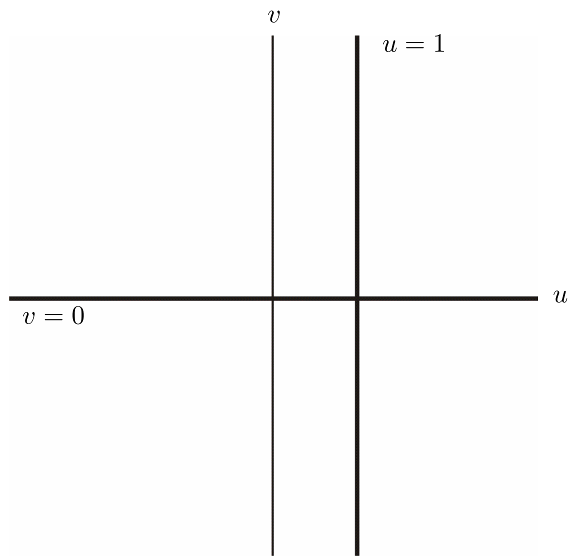
The subject under investigation here is mappings of the complex plane and the particular mapping considered here $\omega \to \omega^2 - 2\omega$ has the property that it can be written in terms of $\omega$ only (i.e. not the complex conjugate $\omega^*$). Such mappings are extremely important in both pure mathematics and also theoretical physics (where, for example, they can be used to solve problems in electrostatics and fluid flow). They have the property that they preserve angles; if we consider the set of lines in the Argand diagram for $\omega$ given by setting the real part of $\omega^2 - 2\omega$ equal to a constant (not necessarily zero) and the set of lines given by setting the imaginary part of $\omega^2 - 2\omega$ equal to a constant, we would find that they intersect at right angles, because the corresponding lines in the mapped Argand diagram meet at right angles (being parallel to the real and imaginary axes). This sort of mapping is called a *conformal transformation*. Conformal transformations were much used in aerofoil theory, because you can use them to straighten out the boundaries of awkward regions of a 2-D plane while preserving the equations that govern the flow of air over the wing; the mathematics is very elegant and is said to have set back aerofoil design many years (other less elegant techniques would have given better results).

**Solution to Question 40**

(i) Let $\omega = u + iv$ (where $u$ and $v$ are real). $\omega^2 - 2\omega$ is real, so $(u+iv)^2 - 2(u+iv)$ is real, i.e.

$$2uv - 2v = 0.$$

Factorising leads to two possibilities: $u = 1$ or $v = 0$. These two lines make up the required locus as shown below.

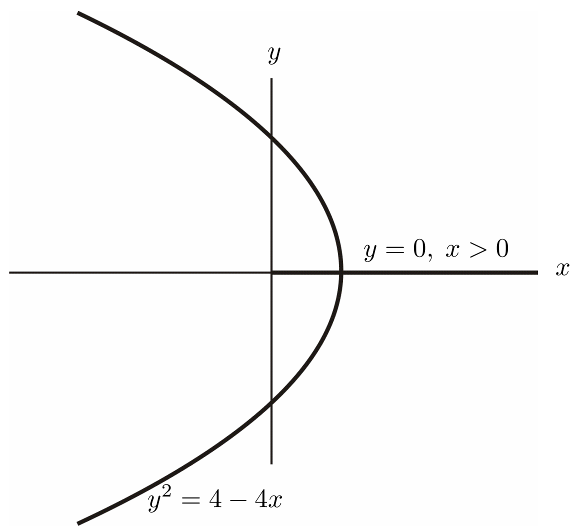



We deal with the two parts of the second locus separately.

If $u = 1$, then $\omega^2 = (1 - v^2) + 2iv$ so $x = (1 - v^2)$ and $y = 2v$. As $v$ varies, this gives the parabola $y^2 = 4 - 4x$.

If $v = 0$, then $x = u^2$ and $y = 0$. As $u$ varies, this gives the non-negative (since $u^2 \geqslant 0$) $x$-axis.

The loci are shown in the sketch below.



(ii) We use the method of part (i). Setting $t = u + iv$, the condition that $t^2 - 2t$ is imaginary translates to

$$u^2 - v^2 - 2u = 0, \tag{$*$}$$

i.e. $(u-1)^2 - v^2 = 1$, which is a (two-branched) hyperbola.
Setting $t^2 = x + iy$ and taking real and imaginary parts gives

$$u^2 - v^2 = p, \qquad 2uv = q.$$

We can eliminate $v$ from these equations using $(*)$:

$$p = u^2 - v^2 = 2u, \qquad q^2 = 4u^2v^2 = 4u^2(u^2 - 2u) = \tfrac{1}{4}p^3(p-4)$$

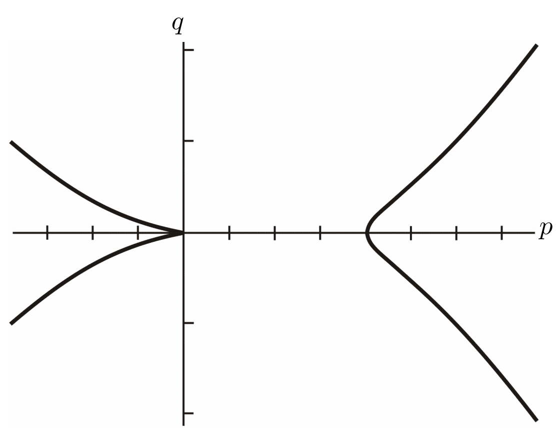
To sketch the quartic curve, note that:

- The curve is symmetrical about the $p$-axis.
- For real $q$, we must have either $p \leqslant 0$ or $p \geqslant 4$.
- For large $p$, the curve approximates to the parabola $q = +p^2/2$ for $q > 0$ and to the parabola $q = -p^2/2$ for $q < 0$.
- Differentiating $q^2 = p^4/4 - p^3$ gives the gradient of the curve:

  $$\frac{\mathrm{d}q}{\mathrm{d}p} = \frac{p^3 - 3p^2}{2q} = \pm\frac{p^3 - 3p^2}{2p\sqrt{p^2/4 - p}} = \pm\frac{\sqrt{p}\,(p-3)}{\sqrt{p-4}}$$

  which shows that the curve is horizontal at $(0,0)$ and vertical at $(4,0)$.

Putting this information together gives the following rough sketch.



**Question 41**

An examination consists of several papers, which are marked independently. The mark given for each paper can be any integer from 0 to $m$ inclusive, and the total mark for the examination is the sum of the marks on the individual papers. In order to make the examination completely fair, the examiners decide to allocate the marks for each paper at random, so that the probability of a given candidate being allocated $k$ marks $(0 \leqslant k \leqslant m)$ for a given paper is $(m+1)^{-1}$.

(i) If there are just two papers, show that the probability of a given candidate receiving a total of $n$ marks is

$$\frac{2m-n+1}{(m+1)^2}$$

for $m \leqslant n \leqslant 2m$. Find the corresponding probability for $0 \leqslant n \leqslant m$.

(ii) If the examination consists of three papers, show that the probability of a given candidate receiving a total of $n$ marks is

$$\frac{6mn-3m^2-2n^2+3m+2}{2(m+1)^3}$$

in the case $m < n \leqslant 2m$. Find the corresponding result for $0 \leqslant n \leqslant m$ and deduce the result for $2m < n \leqslant 3m$.

**Discussion**

There is a low-tech method of tackling this problem, which essentially just involves counting the number of ways of distributing $n$ marks amongst the papers, each distribution being equally probable. You have to do some summations, which are not too messy if you remember that

$$\sum_{s=a}^{b} s = \tfrac{1}{2}(b+a)(b-a+1) \qquad \text{i.e. average term} \times \text{number of terms.}$$

The high-tech method makes use of the probability generating function, but you then have to have some good ideas to fish out the coefficient of $t^n$ from a fairly complicated expression.

The form of the answers is somewhat different for the two methods, so the result given in the question is all multiplied out to give an unpleasant but neutral formula.

There is a helpful symmetry which operates in both parts of the question. There is no difference (at least as far as probabilities are concerned!) in getting one mark and getting no mark at each point where a mark is possible. Therefore, the probability of getting $n$ marks on (e.g.) three papers is the same as the probability of getting $n$ 'non-marks', i.e. $3m-n$ marks. This can be used to infer the $2m \leqslant n \leqslant 3m$ result from the $0 \leqslant n \leqslant m$ result, and to check the result in the symmetrical $m \leqslant n \leqslant 2m$ case (it should remain the same under the transformation $n \to 3m-n$).

**Solution to Question 41**

**Low-tech method.**

Let the mark that the candidate gets from paper 1 be $n_1$, etc, so that $n_1+n_2=n$ for two papers and $n_1+n_2+n_3=n$ for three papers.

(i) If $m\leqslant n\leqslant 2m$, then $n-m\leqslant n_1\leqslant m$ (the first inequality is $n_2\leqslant m$), and the number of different ways of getting a total of $n$ marks is the number of different values for $n_1$, i.e.

$$m-(n-m)+1=(2m-n+1).$$

Each distribution of marks occurs with probability $1/(m+1)^2$, so the required probability it $(2m-n+1)/(m+1)^2$, as given.

If $0\leqslant n\leqslant m$, then $0\leqslant n_1\leqslant n$ and the total number of ways of getting a total of $n$ marks is $(n+1)$. The probability is therefore $(n+1)/(m+1)^2$, which agrees with the previous result if $n=m$.

(ii) For three papers,

$$\begin{aligned}
P\{n\text{ marks}\} &= \sum_k P\{n_1+n_2=k\}\times P\{n_3=n-k\}\\
&= \left[\sum_{k\leqslant m}\frac{k+1}{(m+1)^2}+\sum_{k>m}\frac{2m-k+1}{(m+1)^2}\right]\times\frac{1}{m+1}, \qquad (1)
\end{aligned}$$

using the results of part (i), where the range of the sums over $k$ depend on the value of $n$.

If $m<n\leqslant 2m$ (we use the strict inequality to avoid the upper limit in the second sum below being greater than the lower limit), then $n-m\leqslant n_1+n_2\leqslant n$ and

$$\begin{aligned}
P\{n\text{ marks}\} &= (m+1)^{-3}\left[\sum_{k=n-m}^{m}(k+1)+\sum_{k=m+1}^{n}(2m-k+1)\right]\\
&= (m+1)^{-3}\left[\sum_{s=n-m+1}^{m+1}s+\sum_{k=m}^{2m-n+1}s\right].
\end{aligned}$$

Using the result $\sum\limits_{s=a}^{b}s=\frac{1}{2}(b+a)(b-a+1)$, and tidying up a little, gives the displayed answer.

For $0\leqslant n\leqslant m$, only the first sum of equation (1) is required, so

$$P\{n\text{ marks}\}=(m+1)^{-3}\sum_{k=0}^{n}(k+1)=\frac{(n+1)(n+2)}{2(m+1)^3}.$$

The probability of obtaining $n$ marks is the same as the probability of obtaining $n$ 'non-marks', i.e. $(3m-n)$ marks, so for $2m\leqslant n\leqslant 3m$,

$$P\{n\text{ marks}\}=\frac{(3m-n+1)(3m-n+2)}{2(m+1)^3}.$$

This result is easily also obtainable from the sum (1), only the second term of which is required.

**High-tech method.**

The probability generating function for the mark distribution on one paper is

$$(m+1)^{-1}\big[1+t+t^2+\cdots+t^m\big] = (m+1)^{-1}\frac{1-t^{m+1}}{1-t}.$$

(The coefficient of $t^k$ is the probability of getting $k$ marks.)

(i) For two papers, we need the coefficient of $t^n$ in

$$(m+1)^{-2}\left[\frac{1-t^{m+1}}{1-t}\right]^2 \equiv (m+1)^{-2}(1-2t^{m+1}+t^{2m+2})(1+2t+3t^2+\cdots+(k+1)t^k+\cdots)$$

The third bracket is the binomial expansion for $(1-t)^{-2}$.

If $n \leqslant m$, then only the first term (i.e. 1) of the second bracket contributes, and this must be multiplied by the $t^n$ term from the third bracket. This term has coefficient $n+1$, so the required probability is $(n+1)/(m+1)^2$.

If $n \geqslant m+1$, then there is a contribution from both of the terms in the second bracket; the first is $n+1$ as before, and the second is $(-2)\times$ the coefficient of $t^{n-m-1}$ in the third bracket, which gives $-2(n-m)$. The total is therefore $2m-n+1$, as expected.

(ii) For three papers, we need the coefficient of $t^n$ in

$$\equiv (m+1)^{-3}(1-3t^{m+1}+3t^{2m+2}-t^{3m+3})(1+2.3t/2+3.4t^2/2+\cdots+(k+1)(k+2)t^k/2+\cdots),$$

the last bracket being the series for $(1-t)^{-3}$. If $m<n\leqslant 2m$, we get contributions from both 1 and $-3t^{m+1}$ in the second bracket, so the result is

$$\Big[\frac{n(n+1)}{2}-\frac{3(n-m)(n-m+1)}{2}\Big](m+1)^{-3}.$$

This is another way of expressing the given answer. The other results follow easily.

**Question 42**

One end of a thin inextensible, but perfectly flexible, string of length $l$ and with uniform mass per unit length is held at a point on a smooth table a distance $d$ $(< l)$ away from a small vertical hole in the surface of the table. The string passes through the hole so that a length $l - d$ of the string hangs vertically. The string is released from rest. Assuming that the height of the table is greater than $l$, find the time taken for the end of the string to reach the top of the hole.

**Discussion**

This is a 'variable mass' question. The forces (tension in the string and gravity) act on portions of string whose lengths and hence masses are changing. We cannot therefore use Newton's law in the 'force = mass times acceleration' form.

There are two ways of approaching this problem. One is by conservation of energy, which means that you have to calculate the potential and kinetic energies of each portion of the string. This results in a standard integral of the inverse hyperbolic function sort, which is equivalent (by a small change of variable) to

$$\int \frac{1}{\sqrt{x^2 - a^2}}\, \mathrm{d}x.$$

If you know about hyperbolic functions, you can integrate this by means of the substitution $x = a\cosh\theta$, giving $\cosh^{-1}(x/a)+$ constant. If you do not know about hyperbolic functions, you could look them up (once you understand trig. functions, you should find hyperbolic functions straightforward and interesting) or you could use the following trick. Multiply the top and bottom of the integrand by $x + \sqrt{x^2 - a^2}$ and rearrange:

$$\frac{1}{\sqrt{x^2 - a^2}} = \frac{1}{\sqrt{x^2 - a^2}} \times \frac{x + \sqrt{x^2 - a^2}}{x + \sqrt{x^2 - a^2}} = \frac{\left(x \Big/ \sqrt{x^2 - a^2}\,\right) + 1}{x + \sqrt{x^2 - a^2}}.$$

Now the top is the derivative of the bottom and so can be integrated directly to give a log.

Alternatively, you can use Newton's law in the 'force=rate of change of momentum' form for each portion of the string. This leads to a pair of equations involving the tension in the string. You have to be careful of the signs. Eliminating the tension leads to a simple second differential order equation. You do not need to know about hyperbolic functions to solve this equation (since you can use exponentials instead), but it helps.

Your answer should work for the special cases $d = l$ (when the string will not start moving) and $d = 0$ (when the end is already at the hole). It is also worth checking your solution at various stages for dimensional consistency; this often shows up small algebraic slips.

You might wonder what happens to all the energy when the string lands in a heap on the floor. Presumably, it is all converted into sound (not much of that) and heat (created when the string bends).

**Solution to Question 42**

Let $x$ be the length of the string remaining in contact with the table at time $t$ and let $\rho$ be the mass per unit length. Then $x = d$ initially, at $t = 0$, and $x = 0$ finally, at $t = t_0$ (say). The speed of the string (downwards for the hanging portion) is $-\dot{x}$.

**Method (i): conservation of energy.**

The kinetic energy of the whole string is $\frac{1}{2}(l\rho)\dot{x}^2$. The potential energy is just due to the hanging portion of length $l - x$. The centre of mass of this portion is at a distance $\frac{1}{2}(l - x)$ below the level of the table, so its potential energy is $-\frac{1}{2}(l - x) \times g \times \rho(l - x)$. The initial energy is all potential, being $-\frac{1}{2}\rho(l - d)^2$. The conservation of energy equation is therefore

$$\tfrac{1}{2}(l\rho)\dot{x}^2 - \tfrac{1}{2}\rho g(l - x)^2 = -\tfrac{1}{2}\rho g(l - d)^2.$$

Cancelling $\frac{1}{2}\rho$ and rearranging gives

$$l\dot{x}^2 = g(l - x)^2 - g(l - d)^2, \quad \text{i.e.} \quad \frac{\mathrm{d}x}{\mathrm{d}t} = -(g/l)^{\frac{1}{2}}\sqrt{(l - x)^2 - (l - d)^2}.$$

We have taken the negative square root because $x$ decreases; the equation with the positive square root is just the time reverse (corresponding to the string having been given an initial tug away from the hole).

Separating variables gives

$$(g/l)^{\frac{1}{2}}\int_0^{t_0} \mathrm{d}t = -\int_d^0 \frac{1}{\sqrt{(l - x)^2 - (l - d)^2}}\,\mathrm{d}x.$$

The substitution for this type of integral (see discussion on previous page) is $(l - x) = (l - d)\cosh\theta$, which leads to

$$(g/l)^{\frac{1}{2}} t_0 = -\int_0^{\cosh^{-1}\frac{l}{l-d}} \frac{1}{(l - d)\sinh\theta}\,\big(-(l - d)\sinh\theta\,\mathrm{d}\theta\big) = \int_0^{\cosh^{-1}\frac{l}{l-d}} \mathrm{d}\theta = \cosh^{-1}\frac{l}{l - d}.$$

Thus $t_0 = (l/g)^{\frac{1}{2}}\cosh^{-1}\frac{l}{l-d}$.

**Method (ii): Newton's law**.

Let $T$ be the tension in the string at the point where it passes through the hole. (The tension varies along the string. Consider a length of the string which is in contact with the table. This is being pulled along towards the hole by the tension, so there must be a net force. The tension at the end nearest the hole must be greater than the tension at the other end.)

The portion of the string on the table has mass $\rho x$ and speed towards the hole $-\dot{x}$. The equation of motion is therefore

$$-\frac{\mathrm{d}}{\mathrm{d}t}(\rho x\dot{x}) = T. \qquad (*)$$

For the hanging portion, the mass is $\rho(l-x)$ and the speed upwards is $\dot{x}$. In addition to the tension (acting upwards), there is a downwards force $\rho g(l-x)$. The equation of motion is therefore

$$-\frac{\mathrm{d}}{\mathrm{d}t}\big(\rho(l-x)\dot{x}\big) = -T + \rho g(l-x). \qquad (**)$$

Adding equations $(*)$ and $(**)$, and simplifying, gives

$$l\ddot{x} = g(x-l).$$

We might have written this down straightaway, since it is the equation of motion of the whole string. (The fact that the string is bent is not relevant: the situation is the same as that of a horizontal string which is subject to a force $g\rho(l-x)$.)

One way of solving this is to write it in the form

$$l\ddot{y} = gy,$$

where $y = x - l$ (so that $\ddot{x} = \ddot{y}$). This has solution

$$y = A\cosh\big((g/l)^{\frac{1}{2}}\,t\big) + B\sinh\big((g/l)^{\frac{1}{2}}\,t\big),$$

though it can equally well be written in terms of exponentials. At $t=0$, $\dot{y}=0$, so $B=0$. At $t=0$, $y = d-l$, so the solution is

$$x - l = (d-l)\cosh\big((g/l)^{\frac{1}{2}}\,t\big)$$

At $t = t_0$, $x = 0$, so $-l = (d-l)\cosh\big((g/l)^{\frac{1}{2}}\,t_0\big)$, i.e. $t_0 = (l/g)^{\frac{1}{2}}\cosh^{-1}\frac{l}{l-d}$.

**Question 43**

Let

$$\mathrm{f}(\theta) = \sum_{k=1}^{n} k^{-1} \sin k\theta.$$

By considering the real part of $\sum_{k=1}^{n} \exp(ik\theta)$, show that

$$\mathrm{f}'(\theta) = \frac{\cos(\frac{1}{2}(n+1)\theta)\sin(\frac{1}{2}n\theta)}{\sin(\frac{1}{2}\theta)} = \frac{\sin m\theta}{2\sin(\frac{1}{2}\theta)} - \frac{1}{2},$$

where $m = n + \frac{1}{2}$ and $\frac{1}{2}\theta \neq N\pi$ for any integer $N$.

Show further that, for $0 < \theta_1 < \theta_2 < 2\pi$,

$$\mathrm{f}(\theta_1) - \mathrm{f}(\theta_2) = \frac{\theta_2 - \theta_1}{2} + \frac{\cos m\theta_2}{2m\sin(\frac{1}{2}\theta_2)} - \frac{\cos m\theta_1}{2m\sin(\frac{1}{2}\theta_1)} + \frac{1}{4m}\int_{\theta_1}^{\theta_2} \frac{\cos m\theta \cos(\frac{1}{2}\theta)}{\sin^2(\frac{1}{2}\theta)} \mathrm{d}\theta \qquad (\dagger)$$

and deduce that, for $0 < \theta < 2\pi$,

$$\sum_{k=1}^{\infty} k^{-1} \sin k\theta = \frac{\pi}{2} - \frac{\theta}{2}. \qquad (\ddagger)$$

Deduce further that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots \ .$$

What is the value of the sum in ($\ddagger$) when $2\pi < \theta < 4\pi$? Can the value of this sum in the limit $\theta \to 0$ be inferred from the above results?

**Discussion**

I retrieved this excellent question from the bin after it had been judged too hard for STEP. I suppose it is a bit on the hard side but please don't be put off, because it is certainly interesting. Who would have thought that all the sines in ($\ddagger$) add up to something as simple as $(\pi - \theta)/2$? In fact, who would have thought they summed to anything sensible; after all, if you leave out the sines, you get $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ and this series diverges. (That is, the sum increases without limit as you add on more terms. You can see this by comparing the sum to $n$ terms with the area under the graph of $y = x^{-1}$ from $x = 1$ to $x = n + 1$. A rough sketch shows that the sum represents an area greater than the area under the curve, but the area under the curve tends to infinity as $n$ tends to infinity.)

Although the building blocks for this question are not difficult (differentiating a sum term by term, summing a geometric progression, integrating by parts, investigating terms which tend to zero as $m$ tends to infinity, and finally choosing appropriate values for $\theta_1$ and $\theta_2$), there are hidden depths, with sharks patrolling. For example, you might think that since we are aiming for the infinite series ($\ddagger$), it would have been sensible to define $\mathrm{f}(\theta)$ to be the sum to $k = \infty$, instead of the sum to $k = n$. However, this leads to immediate problems, because while $\mathrm{f}'(\theta)$ would certainly make sense (in fact, ($\ddagger$) shows that $\mathrm{f}'(\theta) = -\frac{1}{2}$), the result of differentiating the sum term by term is a divergent series: the derivative of the sum is not the same (in this case) as the sum of the derivatives. This is the sort of problem addressed by the subject *mathematical analysis* which is encountered in all university courses. Another such problem is the formal proof that the integral in ($\dagger$) tends to zero as $m$ becomes large; here only a rough intuitive discussion is required.

The most important hint (which could apply to any of the questions in this booklet) is to keep firmly in mind the result you are trying to prove when embarking on each step.

**Solution to Question 43**

Differentiating the sum term by term (which is always permitted when there are only a finite number of terms) gives

$$\begin{aligned}
f'(\theta) &= \sum_{k=1}^{n} \cos k\theta = \mathrm{Re}\ \sum_{k=1}^{n} e^{ik\theta} = \mathrm{Re}\ e^{i\theta}\ \frac{1-e^{in\theta}}{1-e^{i\theta}} = \mathrm{Re}\ \frac{e^{in\theta/2}-e^{-in\theta/2}}{e^{i\theta/2}-e^{-i\theta/2}}\ e^{i(n+1)\theta/2} \\
&= \mathrm{Re}\ \frac{\sin(\frac{1}{2}n\theta)}{\sin(\frac{1}{2}\theta)}\ e^{i(n+1)\theta/2} = \frac{\sin(\frac{1}{2}n\theta)}{\sin(\frac{1}{2}\theta)}\ \cos(\tfrac{1}{2}(n+1)\theta) = \frac{\sin m\theta}{2\sin(\frac{1}{2}\theta)} - \frac{1}{2}.
\end{aligned}$$

Here, Re means that only the real part of the expression is used. For the second equality, we use $e^{ik\theta} = \cos k\theta + i\sin k\theta$, the real part of which is $\cos k\theta$. The third equality comes from the standard formula for the sum of a geometric progression. The rearrangement of the exponentials for the fourth equality is not very obvious, but is driven by the form of the expression we are aiming for. For the fifth equality, we use the identity $\sin x = (e^{ix} - e^{-ix})/(2i)$. The real part of $e^{i(n+1)\theta/2}$ is $\cos(\frac{1}{2}(n+1)\theta)$, so the sixth equality follows immediately. The seventh equality requires a standard trig. identity in the form $2\cos A\sin B = \sin(A+B) - \sin(A-B)$.

To obtain (†), we integrate the previous result:

$$\int_{\theta_1}^{\theta_2} f'(\theta)\,d\theta = \int_{\theta_1}^{\theta_2} \left(\frac{\sin m\theta}{2\sin(\frac{1}{2}\theta)} - \frac{1}{2}\right) d\theta \quad \text{i.e.} \quad f(\theta_2) - f(\theta_1) = \int_{\theta_1}^{\theta_2} \frac{\sin m\theta}{2\sin(\frac{1}{2}\theta)}\,d\theta - \frac{1}{2}(\theta_2 - \theta_1).$$

Now integration by parts (integrate $\sin m\theta$, differentiate $\big(\sin\frac{1}{2}\theta\big)^{-1}$) gives (†) with an overall minus sign.

Next we need to see what happens when $m$ tends to infinity. Obviously the two terms in (†) with $m$ in the denominator and $\cos m\theta$ in the numerator tend to zero (since $-1 \leqslant \cos m\theta \leqslant 1$). For the integral in (†), we have

$$\left|\frac{1}{4m}\int_{\theta_1}^{\theta_2} \frac{\cos m\theta\cos(\frac{1}{2}\theta)}{\sin^2(\frac{1}{2}\theta)} d\theta\right| \leqslant \frac{1}{4m}\int_{\theta_1}^{\theta_2} \frac{1}{\sin^2(\frac{1}{2}\theta)} d\theta = \frac{\cot(\frac{1}{2}\theta_1) - \cot(\frac{1}{2}\theta_2)}{2m}$$

(replacing each of the cosines by +1) so this term also tends to zero as $m \to \infty$. A less rigorous argument here would be perfectly in order.

Finally, we take take the limit $m \to \infty$ and set $\theta_1 = \theta$, $\theta_2 = \pi$ to obtain (‡). Note that $f(\pi) = 0$, since all of the sines in the original sum are zero when $\theta = \pi$. Setting $\theta = \pi/2$ in (‡) gives the (nice but useless, since it needs a million terms to get seven figure accuracy) formula for $\pi/4$.

The sum in (‡) is unchanged when $\theta \to \theta + 2\pi$, so it represents a periodic, sawtooth-shaped, graph from which the value for $2\pi < \theta < 4\pi$ can be inferred. Alternatively, since the derivation of (†) also works if $2\pi < \theta < 4\pi$, we could take $\theta_1 = \theta$, $\theta_2 = 3\pi$ in the final step. The value of the sum when $\theta = 0$ is obviously 0, which is not the same as the value of $(\pi - \theta)/2$. The reason for this is that the sum represents a *discontinuous* function (the sawtooth) which jumps in value at $\theta = 0$.

**Further discussion**

The series in (‡) is called a Fourier series after the French mathematician Joseph Fourier. More or less any periodic function can be expanded in terms of sines and cosines, and this expansion provides a good way of solving the equation that determines the temperature of (say) a bar which is heated at one end. Fourier served as an administrator in Egypt under Napoleon, and was stranded there when Napoleon rushed back to Paris, abandoning his army. The two consequences of this apparent misfortune were that Fourier developed a life-long interest in Egyptian culture and an obsessive need to keep warm. His thirteen volumes on Egyptian culture marked the foundation of the modern study of Egyptology and his treatise on the theory of heat led to a completely new approach to the subject (but probably did not keep him warm).