**I/2006/1** - Find the integer, n, that satisfies $n^2 < 33\,127 < (n+1)^2$. Find also a small integer m such that $(n+m)^2 - 33\,127$ is a perfect square. Hence express 33 127 in the form pq, where p and q are integers greater than 1.

By considering the possible factorisations of 33 127, show that there are exactly two values of $m$ for which $(n+m)^2 - 33\,127$ is a perfect square, and find the other value.

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**Something like this will do for a solution. See below for further discussion**

By inspection $182^2 = 33\,124$, so the required $n = 182$.

$183^2 - 33\,127 = 183^2 - 182^2 - 3 = 362$

$184^2 - 33\,127 = 184^2 - 183^2 + 362 = 729 = 27^2$

so the required $m = 2$.

$184^2 - 33\,127 = 27^2 \implies 33\,127 = (184+27)(184-27) = 211 \times 157$

Each equation $(182+m)^2 - 33\,127 = r^2$ corresponds to a factorisation $33\,127 = (182+m+r)(182+m-r)$

so there are exactly two positive values of m for which $(n+m)^2 - 33\,127$ is a perfect square if 33 127 has only two factorisations.

$33\,127 = 211 \times 157 = 33\,127 \times 1$; since, by inspection, 211 and 157 are prime, there are only two factorisations. To find the required $m$ for the $33\,127 \times 1$ factorisation:

$182 + m + r = 33\,127$

$182 + m - r = 1$

so $2m = 33\,128 - 364 = 32\,764$, and $m = 16\,382$

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**Further discussion**

"By inspection" is a phrase often used in maths, meaning: just check out the calculation, or the diagram - enough said!

How do we know to try $182^2$? First try some very easy squares. $100^2 = 10\,000$ and $200^2 = 40\,000$. Those calculations show $n$ must be between 100 and 200, and closer to 200.

If you have a reasonable memory for numbers, it is worth learning the square numbers up to at least $19^2$, just as it is good to learn the streets of your neighbourhood so you can find your way round by memory without need for satnav or Google Maps. Also worth learning the cube numbers up to say $12^3$, and the powers of 2 up to say $2^{10}$. Then you will know straight off that $18^2 = 324$ and $180^2 = 32\,400$.

However, some brilliant mathematicians have been rubbish at mental arithmetic. Eduard Kummer, 1810-1893, is the famous example. Maybe you're another example. What then?

Work down from $200^2$ in tens, using difference of two squares.

$200^2 - 190^2 = 100 \times (20^2 - 19^2) = 100 \times (20 + 19)(20 - 10) = 3\,900$, so $190^2 = 36\,100$ - too big

$190^2 - 180^2 = 100 \times (19^2 - 18^2) = 3\,700$, so $180^2 = 32\,400$ - too small

Now work up from $180^2$ in units

$181^2 - 180^2 = (181 + 180)(181 - 180) = 361$, so $181^2 = 32\,761$ - too small

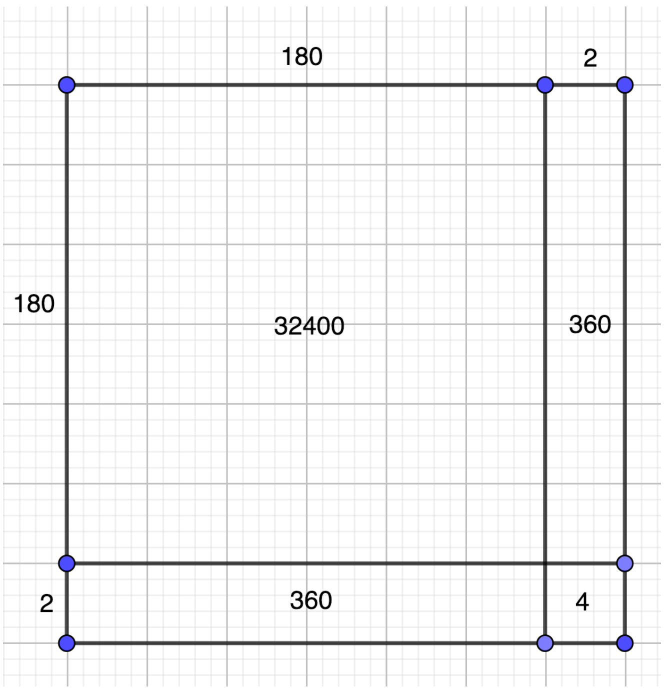
$182^2 - 181^2 = 363$, so $181^2 = 33\,124$, so $n = 182$ is what we want.

This method is easier and more reliable than long multiplication.

Another way of working out the necessary squares, easier than long multiplication, is:

$182^2 = (180 + 2)^2 = 180^2 + 2 \times 180 \times 2 + 2^2 = 32\,400 + 720 + 4 = 33\,124$

Or essentially the same thing in graphic form




How do we know that 362 is not a square number, but 729 is? Square numbers must have units digit 0, 1, 4, 5, 6, or 9. 362 can't be a square number, because its last digit is 2. 729 might be a square number. To find out, work on its prime factorisation: $729 = 3 \times 243 = 3^2 \times 81 = 3^2 \times 9^2 = 27^2$.

For all odd numbers, each factorisation corresponds to an expression as difference-of-squares. This is called *Fermat factorisation.*

If $N = pq$, then $N = \left(\frac{p+q}{2}\right)^2 - \left(\frac{p-q}{2}\right)^2$, and if $N = b^2 - c^2$, then $N = (b+c)(b-c)$. This works because if N is odd, $p$ and $q$ must be odd, and so $\left(\frac{p+q}{2}\right)$ and $\left(\frac{p-q}{2}\right)$ are whole numbers.

So in the second part of the question, "exactly two values of $m$" (really, they should have written: exactly two *positive* values of $m$) is equivalent to "exactly two factorisations".

Every number N has at least one factorisation, $1 \times N$, even if it is prime. If 211 or 157 were not prime, e.g. $211 = jk$, then there would be at least a third factorisation, $j \times 157k$, so what we have to do is prove 211 and 157 are prime.

To do that, we have to prove e.g. that no number bigger than 1 and smaller than 211 divides into 211. But we don't have to check all 209 numbers between 2 and 210.

First, we can limit ourselves to *prime* numbers, because if any number $j > 1$ divides into 211, then some prime factor of $j$ also divides into 211.

Then, we can limit ourselves to prime numbers $p < \sqrt{211}$, because if a prime number $q > \sqrt{211}$ divides into 211, then $211 = qr$ where $r < \sqrt{211}$. For example, if 17 divided into 211, then we would have $17 \times r = 211$, with $r$ smaller than $\sqrt{211}$.

So in fact we need only check whether $2, 3, 5, 7, 11, 13$ divide exactly into 211. And for 157 we need only check whether $2, 3, 5, 7, 11$ divide into it.

The idea here is called Eratosthenes' Sieve:

http://demonstrations.wolfram.com/SieveOfEratosthenes/.

The checking is not hard even for numbers in the thousands. In August 2017, I had a "grand mal" epileptic seizure while working in the maths, science, and engineering library at the University of Queensland.

In hospital afterwards nurses kept checking whether I was in a coma, had had a stroke, had broken my neck, etc., and to do so they would ask me what year it was.

I convinced the nurses by answering: "2017, and 2017 is a prime number". I had to check prime factors up to 43 in my head. Not hard. To check 43, for example, see that $43 \times 50 = 2150$, so 43 divides exactly into 2017 only if it divides exactly into 133. Which it doesn't.

2019 was not prime (obviously: but why obviously?). 2021 is not prime, either: $2021 = 43 \times 47$. Nor 2023, which is $7 \times 17^2$. Nor 2025 (again, obviously, but why obviously?)

But 2027 is. Check it out? Again, check possible prime factors up to 43.