STEP I 2011 WORKED ANSWERS

① (i) $\frac{a}{x} + \frac{b}{y} = 1$

so $-\frac{a}{x^2} - \frac{b}{y^2}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-ay^2}{bx^2}$

Gradient of line & curve same $\Rightarrow -\frac{a}{b} = \frac{-aq^2}{bp^2}$

$\Rightarrow \frac{q^2}{p^2} = 1 \Rightarrow p = \pm q$

(p,q) on both line and curve $\Rightarrow$

$$ap \pm bp = 1 \Rightarrow (a \pm b) = \frac{1}{p}$$

$$\frac{a}{p} \pm \frac{b}{p} = 1 \Rightarrow (a \pm b) = p$$

$$\therefore (a \pm b)^2 = 1$$

Line $\perp$ curve $\Rightarrow -\frac{a}{b} \cdot \left(+\frac{aq^2}{bp^2}\right) = -1$

$$\therefore \frac{a^2}{b^2} = \frac{p^2}{q^2} \Rightarrow \frac{a}{p} = \pm\frac{b}{q} \quad [1]$$

(p,q) on both line and curve $\Rightarrow ap + bq = 1 \quad [2]$

and $a/p - b/q = 1 \quad [3]$

$[1] + [3] \Rightarrow \frac{a}{p} = -\frac{b}{q} = \frac{1}{2}$

$\Rightarrow \frac{q}{p} = -\frac{b}{a}$

$[2] \times [3] \Rightarrow a^2 - b^2 + ab\left(\frac{q}{p} - \frac{p}{q}\right) = 1$

$\Rightarrow a^2 - b^2 + ab\left(\frac{a}{b} - \frac{b}{a}\right) = 1$

$\Rightarrow a^2 - b^2 = \frac{1}{2}$

(2) $$E = \int_0^1 \frac{e^x \, dx}{1+x}$$

Let $I = \int_0^1 \frac{x e^x}{1+x} \, dx$

$$I = \int_0^1 e^x \, dx - \int_0^1 \frac{e^x}{1+x} \, dx$$

$$= e - 1 - E$$

Let $J = \int_0^1 \frac{x^2 e^x}{1+x} \, dx = \int_0^1 e^x \left(x - \frac{x}{1+x} \right) dx$

$$= \int_0^1 x e^x \, dx - I$$

$$\int_0^1 x e^x \, dx = \left[x e^x \right]_0^1 - \int_0^1 e^x \, dx = e - (e-1) = 1$$

So $J = 1 - I = 2 - e + E$

(i) $K = \int_0^1 \frac{\exp\left(\frac{1-x}{1+x}\right)}{1+x} \, dx$

$$\frac{1-x}{1+x} = -1 + \frac{2}{1+x}$$

Let $y = \frac{1-x}{1+x}$ and $\frac{1}{1+x} = \frac{1}{2}(y+1)$

$$dy = -\frac{2}{(1+x)^2} \, dx = -(y+1) \frac{1}{(1+x)} \, dx$$

$$K = -\int_1^0 \exp(y) \frac{dy}{y+1} = E$$

③ $\sin\left(\frac{\pi}{3}-\theta\right)\sin\left(\frac{\pi}{3}+\theta\right) = \frac{1}{2}\left[\cos(2\theta) - \cos\frac{2\pi}{3}\right]$

$= \frac{1}{2}\cos 2\theta + \frac{1}{4}$

$\therefore 4\sin\theta\sin\left(\frac{\pi}{3}-\theta\right)\sin\left(\frac{\pi}{3}+\theta\right)$

$= 2\sin\theta\cos 2\theta + \sin\theta$

$2\sin\theta\cos 2\theta = \sin 3\theta - \sin\theta$

$\therefore$ expression $= \sin 3\theta$ as required.

Differentiating the equation and then dividing by it

$\left(\cot\theta - \cot\left(\frac{\pi}{3}-\theta\right) + \cot\left(\frac{\pi}{3}+\theta\right)\right) = 3\cot 3\theta$

Let $\theta = \frac{1}{9}\pi$, then $\cot 3\theta = \frac{\cos \pi/3}{\sin \pi/3} = \frac{1}{\sqrt{3}}$

So $\cot\frac{1}{9}\pi - \cot\frac{2}{9}\pi + \cot\frac{4}{9}\pi = \sqrt{3}$

Set $\theta = \frac{\pi}{6} - \varphi$ in original equation

$\Rightarrow 4\sin\left(\frac{\pi}{6}-\varphi\right)\sin\left(\frac{\pi}{6}+\varphi\right)\sin\left(\frac{\pi}{2}-\varphi\right) = \sin\left(\frac{\pi}{2}-3\varphi\right)$

$\Rightarrow 4\cos\left(\frac{\pi}{3}+\varphi\right)\cos\left(\frac{\pi}{3}-\varphi\right)\cos\varphi = \cos 3\varphi$

Put $\varphi = \theta$ and divide by original equation

$\cot\theta\ \cot\left(\frac{\pi}{3}-\theta\right)\cot\left(\frac{\pi}{3}+\theta\right) = \cot 3\theta$

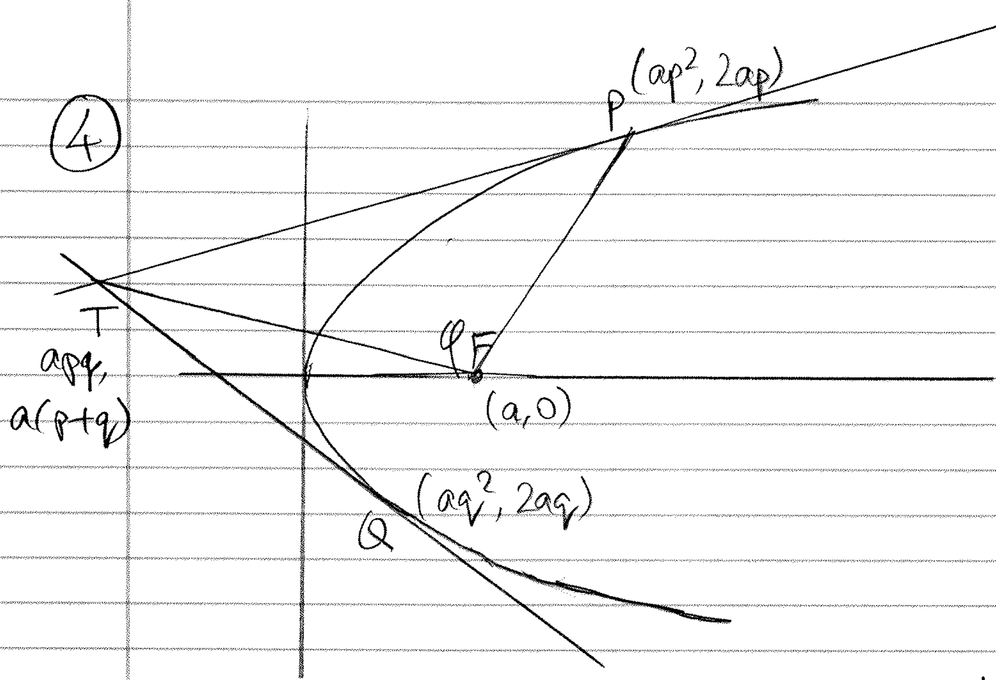
Differentiating this equation and then dividing by it and noting $-\frac{\operatorname{cosec}^2\theta}{\cot\theta} = -\frac{1/\sin^2\theta}{\cos\theta/\sin\theta} = -\frac{1}{\sin\theta\cos\theta} = -\frac{1}{2}\operatorname{cosec} 2\theta$

$\operatorname{cosec} 2\theta - \operatorname{cosec}\left(\frac{2\pi}{3}-2\theta\right) + \operatorname{cosec}\left(\frac{2\pi}{3}+2\theta\right) = 3\operatorname{cosec} 6\theta$

Put $\theta = \frac{\pi}{18}$

$\operatorname{cosec}\frac{\pi}{9} - \operatorname{cosec}\frac{5\pi}{9} + \operatorname{cosec}\frac{7\pi}{9} = \frac{3}{\sin\frac{\pi}{3}} = 2\sqrt{3}$

4



At general point $y = 2at$, $x = at^2$ $\Longrightarrow$ $dy/dt = 2a$, $dx/dt = 2at$

$$\Longrightarrow \frac{dy}{dx} = \frac{1}{t}$$

$\therefore$ Tangent is $(y - 2at) = \frac{1}{t}(x - at^2)$

So for meeting point of tangents at P and Q

$$y - 2ap = \frac{1}{p}(x - ap^2) \qquad [1]$$

$$y - 2aq = \frac{1}{q}(x - aq^2) \qquad [2]$$

$$[1] - [2] \Longrightarrow 2a(q-p) = \left(\frac{q-p}{pq}\right)x + a(q-p)$$

$$\therefore x = apq$$

$$p[1] - q[2] \Longrightarrow (p-q)y - 2a(p^2-q^2) = -a(p^2-q^2)$$

$$\therefore y = a(p+q)$$

Cosine rule in triangle TFP, eliminating a^2 factor

$$(p^2-pq)^2 + (p-q)^2 = (p^2-1)^2 + 4p^2 + (pq-1)^2 + (p+q)^2$$
$$-2\cos\varphi\sqrt{((p^2-1)^2+4p^2)((pq-1)^2+(p+q)^2)}$$

$$(p-q)^2(p^2+1) = (p^2+1)^2 + p^2q^2 + p^2 + q^2 + 1$$
$$-2\cos\varphi\,(p^2+1)\sqrt{(p^2+1)(q^2+1)}$$

4 contd

$$(p-q)^2 = p^2+1 + q^2+1 - 2\cos\phi\sqrt{(p^2+1)(q^2+1)}$$

$$\therefore \cos\phi = \frac{pq+1}{\sqrt{(p^2+1)(q^2+1)}}$$

Since this is symmetrical in p and q, and $\hat{TFP}$ and $\hat{TFQ}$ are both between 0 and π

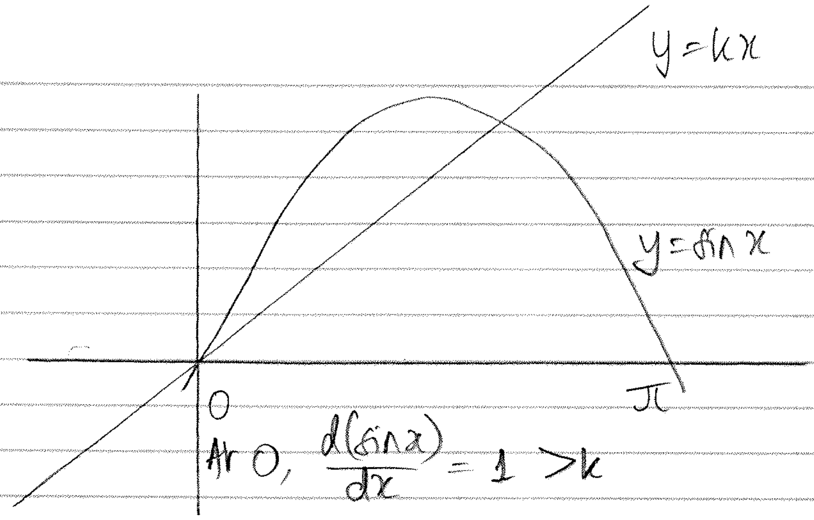
$$\hat{TFP} = \hat{TFQ}$$

$\therefore$ TF bisects $\hat{PFQ}$

This can also be proved more neatly by geometry

bit.ly/parabola7.

(5)



At O, $\frac{d(\sin x)}{dx} = 1 > k$

As long as $k < 1$, $y = kx$ is below $y = \sin x$ for some range in $0 < x < \pi$

As long as $k > 0$, $y = kx$ is above $y = \sin x$ for some range in $0 < x < \pi$

Slope of $y = \sin x$ monotone decreasing for $0 < x < \pi$

$\therefore$ after first intersection slope of $y = \sin x < k$,

$\therefore$ no second intersection can follow for $0 < x < \pi$.

$$I = \int_0^{\alpha} (\sin x - kx)\, dx + \int_{\alpha}^{\pi} (kx - \sin x)\, dx$$

$$= \left[-\cos x - \tfrac{1}{2}kx^2\right]_0^{\alpha} + \left[\cos x + \tfrac{1}{2}kx^2\right]_{\alpha}^{\pi}$$

$$= -\cos\alpha - \tfrac{1}{2}k\alpha^2 - \cos\alpha - \tfrac{1}{2}k\alpha^2$$
$$+1 - 1 + \tfrac{1}{2}k\pi^2$$

$$= \tfrac{1}{2}k\pi^2 - 2\cos\alpha - k\alpha^2$$

substituting $k = \frac{\sin\alpha}{\alpha}$

$$I = \frac{\pi^2 \sin\alpha}{2\alpha} - \alpha\sin\alpha - 2\cos\alpha$$

(5) contd

$$\frac{dI}{d\alpha} = \frac{\pi^2 \cos\alpha}{2\alpha} - \frac{\pi^2 \sin\alpha}{2\alpha^2} - \sin\alpha - \alpha\cos\alpha + 2\sin\alpha$$

$$= \left(\frac{\pi^2}{2\alpha^2} - 1\right)\left(\alpha\cos\alpha - \sin\alpha\right)$$

So $\frac{dI}{d\alpha} = 0$ when $\tan\alpha = \alpha$ (but, by inspecting the graph, that's never true)

or when $\frac{\pi^2}{2\alpha^2} = 1$

ie. $\alpha = \frac{\pi}{\sqrt{2}}$

At that α, $I = \frac{\pi^2 \sin\left(\pi/\sqrt{2}\right)}{\pi\sqrt{2}} - \frac{\pi}{\sqrt{2}} \sin\frac{\pi}{\sqrt{2}} - 2\cos\frac{\pi}{\sqrt{2}}$

$$= -2\cos\frac{\pi}{\sqrt{2}}$$

When k is almost 0, I is almost 2

When k is almost 1, I is almost $\frac{1}{2}\pi^2 - 2 > 2$

$\therefore$ on both sides $I > -2\cos\frac{\pi}{\sqrt{2}}$

$\therefore -2\cos\frac{\pi}{\sqrt{2}}$ is minimum value of I.

(6) Coeff^t of $x^r = \frac{-3.-4.-5.\ldots(-r-2)}{1.2.3.\ldots r}(-1)^r$

$= \frac{1}{2}(r+1)(r+2)$

Coeff^t of x^r in expansion of $(1-x+2x^2)(1-x)^{-3}$

$=$ Coeff^t of x^r $-$ coeff^t of x^{r-1} $+$ $2\times$coeff^t of x^{r-2}

$= \frac{1}{2}(r+1)(r+2) - \frac{1}{2}r(r+1) + (r-1)r$

$= r^2+1$

Series $= 1 + \frac{1^2+1}{2} + \frac{2^2+1}{4} + \frac{3^2+1}{8} + \frac{4^2+1}{16} + \ldots$

$\ldots + \frac{r^2+1}{2^r} + \ldots$

$= \frac{1-x+2x^2}{(1-x)^3}$ when $x = \frac{1}{2}$

$= \frac{1-\frac{1}{2}+\frac{1}{2}}{1/8} = 8$

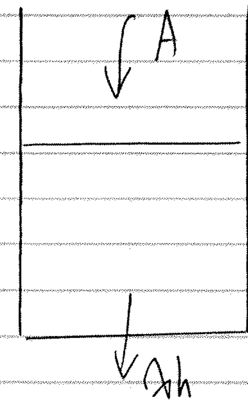
(ii) Cited series $= 1 + \frac{2^2}{2} + \frac{3^2}{4} + \frac{4^2}{8} + \frac{5^2}{16} + \frac{6^2}{32} + \frac{7^2}{64} + \ldots$

$= 2\left(1 + \frac{1^2+1}{2} + \frac{2^2+1}{4} + \frac{3^2+1}{8} + \ldots\right)$

$-2\left(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots\right)$

$= 16 - 4 = 12$

7 Assume to reduce clutter that cross-section = 1



Then $\frac{dh}{dt} = A - \lambda h$ for some const $\lambda > 0$

If $\frac{dh}{dt} = 0$ when $h = \alpha^2 H$

then $A = \lambda \alpha^2 H$

and $\frac{dh}{dt} = \lambda(\alpha^2 H - h)$

Let $\eta = \frac{h}{\alpha^2 H}$, then $\frac{d\eta}{dt} = \lambda(1-\eta)$

so $-\ln(1-\eta) = \lambda t + \text{const.}$

$h = H$, so $\eta = \frac{1}{\alpha^2}$ when $t = 0 \Rightarrow \text{const} = -\ln\left(1 - \frac{1}{\alpha^2}\right)$

When $h = \alpha H$, or $\eta = \frac{1}{\alpha}$

$$-\ln\left(1-\frac{1}{\alpha}\right) = \lambda t - \ln\left(1 - \frac{1}{\alpha^2}\right)$$

$$\therefore \lambda t = \ln\left(1 + \frac{1}{\alpha}\right)$$

For large α, $\ln\left(1+\frac{1}{\alpha}\right) \simeq \frac{1}{\alpha}$

$$\therefore \lambda t \simeq \frac{1}{\alpha}$$

(7) contd

Under the new conditions

$$\frac{dh}{dt} = \lambda\left(\alpha\sqrt{H} - \sqrt{h}\right) \quad \text{for a new constant } \lambda$$

Let $\dfrac{h}{\alpha^2 H} = \eta$

Then $\alpha\sqrt{H}\,\dfrac{d\eta}{dt} = \lambda\left(1 - \sqrt{\eta}\right)$

$$\frac{d\eta}{1-\eta^{\frac{1}{2}}} = \frac{\lambda}{\alpha\sqrt{H}}$$

Let $y = \eta^{1/2}$, so $y^2 = \eta$ and $2y\,dy = d\eta$

$$\frac{2y\,dy}{1-y} = \frac{\lambda}{\alpha\sqrt{H}}$$

$$\left(-2 + \frac{2}{1-y}\right)dy = \frac{\lambda}{\alpha\sqrt{H}}$$

$$-2y - 2\ln(1-y) = \frac{\lambda t}{\alpha\sqrt{H}} + \text{const.}$$

Since $h = H$, so $\eta = \dfrac{1}{\alpha^2}$ and $y = \dfrac{1}{\alpha}$ when $t = 0$

$$\text{const} = -\frac{2}{\alpha} - 2\ln\left(1 - \frac{1}{\alpha}\right)$$

We want t for $h = \alpha H$, ie. $\eta = \dfrac{1}{\alpha}$, $y = \dfrac{1}{\sqrt{\alpha}}$

$$\frac{\lambda t}{\alpha\sqrt{H}} - \frac{2}{\alpha} - 2\ln\left(1 - \frac{1}{\alpha}\right) = -\frac{2}{\sqrt{\alpha}} - 2\ln\left(1 - \frac{1}{\sqrt{\alpha}}\right)$$

$$\frac{\lambda t}{\alpha\sqrt{H}} = 2\ln\left(1 + \frac{1}{\sqrt{\alpha}}\right) + \frac{2}{\alpha} - \frac{2}{\sqrt{\alpha}}$$

$$\therefore \lambda t = 2\sqrt{H}\left(1 - \sqrt{\alpha} + \alpha\ln\left(1 + \frac{1}{\sqrt{\alpha}}\right)\right)$$

(7) <u>contd</u>

If α is large

$$\ln\left(1+\frac{1}{\sqrt{\alpha}}\right) \simeq \frac{1}{\sqrt{\alpha}} - \frac{1}{2\alpha}$$

$$\therefore \lambda t \simeq 2\sqrt{H}\left(1-\sqrt{\alpha}+\sqrt{\alpha}-\frac{1}{2}\right) \simeq \sqrt{H}$$

(8) (i) $m^3 = n^3 + n^2 + 1$ $\quad (*)$

for all n, $n^2 \geq 0$ $\therefore$ $m^3 > n^3$

Since cubic is monotone increasing, $m > n$.

$m < (n+1) \iff m^3 < n^3 + 3n^2 + 3n + 1$

$\iff n^3 + n^2 + 1 < n^3 + 3n^2 + 3n + 1$

$\iff 0 < 2n^2 + 3n$

$2n^2 + 3n = 2n\left(n + \frac{3}{2}\right)$, so is positive if $n < -\frac{3}{2}$

or if $n > 0$

So for all values of n outside $-3/2 \leq n \leq 0$

$n < m < n+1$

So if $(*)$ is true and m and n are integers

Either $n = -1$, and then $m = 1$

or $n = 0$, and then $m = 1$ again

or $n < m < n+1$, which is impossible if m and n are integers.

(ii) $(\ddagger)$ $p^3 = q^3 + 2q^2 - 1 \Rightarrow q = 0$ (and $p = -1$)

or $p > q$

$(\ddagger)$ also $\Rightarrow p^3 = (q+1)^3 - q^2 - 3q - 2$

$= (q+1)^3 - (q+2)(q+1)$

So if $q = -1$, $p = 0$. If $q = -2$, $p = -1$

otherwise if $q > -1$, $p^3 < (q+1)^3$, so $q < p < (q+1)$

which is impossible for integer p, q. And if $q < -2$, ditto.

So only solutions are $p = 0, q = -1$

$p = -1, q = -2$

$p = -1, q = 0$

STEP I/2011/9

The trajectory is $y = Bx - Ax^2$ for some B, A

Since $\frac{\mathrm{d}y}{\mathrm{d}x} = \tan\theta$ at $x = 0$, $B = \tan\theta$, and so

$$d_2 = \tan\theta d_1 - Ad_1^2 \qquad [1]$$

$$d_1 = \tan\theta d_2 - Ad_2^2 \qquad [2]$$

Eliminating A, $d_2^3 - \tan\theta d_1 d_2^2 = d_1^3 - \tan\theta d_1^2 d_2$

So $\tan\theta = \frac{d_1^2+d_1 d_2+d_2^2}{d_1 d_2}$

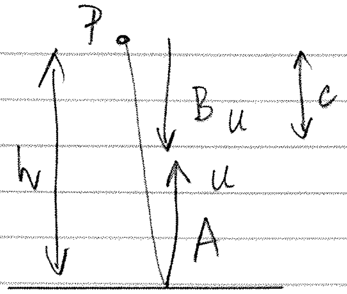
Range = $\frac{\tan\theta}{A}$

[1] and [2] $\implies A = \frac{(d_1-d_2)(\tan\theta+1)}{d_1^2-d_2^2} = \frac{\tan\theta+1}{d_1+d_2}$

$\tan\theta + 1 = \frac{(d_1+d_2)^2}{d_1 d_2}$, so $A = \frac{(d_1+d_2)}{d_1 d_2}$

Range = $\frac{\tan\theta}{A} = \frac{d_1^2+d_1 d_2+d_2^2}{d_1+d_2}$

(10)



If the bounce on the plane is perfectly elastic, then the speed of A at any height bouncing back up = its speed when dropping down at the same height = speed of B at that height.

After first collision, let speeds of A and B be V_A and V_B upwards

then $$Mu - mu = MV_A + mV_B \quad [1]$$

$$2u = V_B - V_A \quad [2]$$

$$[1] - m[2] \Rightarrow Mu - 3mu = (M+m)V_A$$

If $M > 3m$, then LHS of this equation > 0

$\therefore$ RHS > 0

$\therefore$ $V_A > 0$ and both particles move up.

Ball will rise above height h after the first collision by $\frac{v_B{}^2 - u^2}{2g}$, i.e. by an amount proportional to its gain in KE from the collision.

Elastic collision $\Rightarrow v_A = v_B - 2u$

Conservation of momentum $\Rightarrow Mv_A + mv_B = (M-m)u$

$\Rightarrow (M+m)v_B - 2Mu = (M-m)u$

$\Rightarrow (M+m)(v_B - u) = 2(M-m)u$

$\Rightarrow (M+m)(v_B + u) = 4Mu$

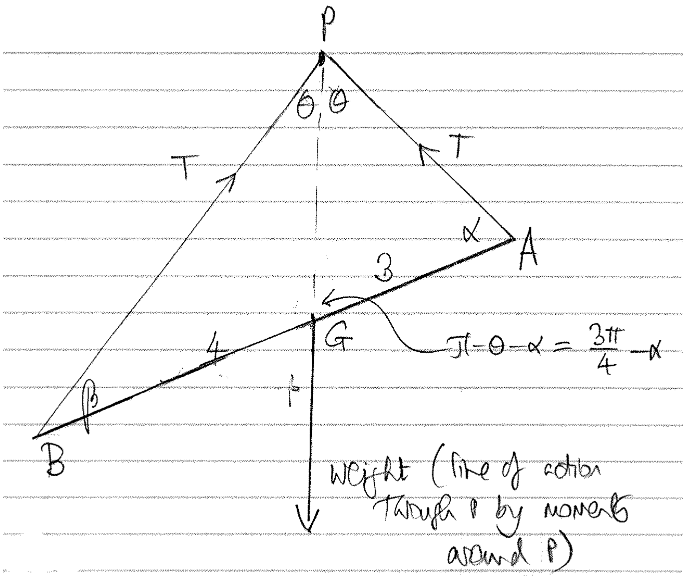
Multiply, $(M+m)^2(v_B{}^2 - u^2) = 8M(M-m)u^2$

Therefore B rises $\frac{4M(M-m)u^2}{(M+m)^2 g}$ above height h.

STEP 1 2011 Q.11

Martin Thomas

May 27, 2017



Peg smooth $\implies$ tension T is the same on both parts of the string, otherwise the string would slip on the peg.

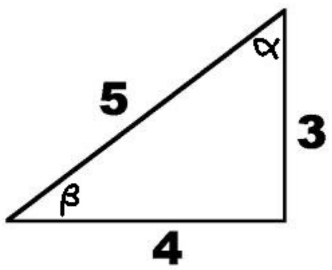
Moments around P $\implies$ G is vertically below P, so that the line of action of the weight passes through P, otherwise the rod would rotate around P

Resolving forces horizontally $\implies$ the two angles marked θ are equal, or else the rod would move sideways

Moments around G $\implies$ 4 sin β = 3 sin α

$\cos\beta = \frac{4}{5} \implies \sin\beta = \frac{3}{5} \implies \sin\alpha = \frac{4}{5}$

So α and β are as shown below, and $2\theta = \frac{\pi}{2}$



In the right-angle triangle BPA, AP $= 7$ sin β and BP $= 7$ sin α

So AP $= \frac{21}{5}$, BP $= \frac{28}{5}$, length of string $= \frac{49}{5}$

Angle between rod and horizontal $= \frac{\pi}{2} - (\frac{3\pi}{4} - \alpha) = \alpha - \frac{\pi}{4}$

so tan (angle) $= \frac{\tan\alpha - \tan\frac{\pi}{4}}{1+\tan\alpha\tan\frac{\pi}{4}} = \frac{\frac{4}{3}-1}{1+\frac{4}{3}} = \frac{1}{7}$

(12) (i) If $n=1$, then I'm good unless the £2 person comes first, which has $P = \frac{n}{n+m} = \frac{1}{m+1}$

$$\therefore P(\text{good}) = \frac{m}{m+1}$$

(ii) Let 0 represent £2 coin, 1 represent £1. Options for first three are

~~0 0 0~~

0	0	1	✗	
0	1	0	✗	
0	1	1	✗	
1	0	0	✗	
1	0	1	✓	$P = \frac{m}{m+2} \cdot \frac{2}{m+1} \cdot \frac{m-1}{m}$
1	1	0	✓	$P = \frac{m}{m+2} \cdot \frac{m-1}{m+1} \cdot \frac{2}{m}$
1	1	1	✓	$P = \frac{m}{m+2} \cdot \frac{m-1}{m+1} \cdot \frac{m-2}{m}$

✗ denotes no good, ✓ denotes good

$$\text{Total } P(\text{good}) = \frac{2m^2-2m + 2m^2-2m + m^3-3m^2+2m}{m(m+1)(m+2)}$$

$$= \frac{m^3+m^2-2m}{m(m+1)(m+2)} = \frac{(m+2)(m-1)m}{m(m+1)(m+2)} = \frac{m-1}{m+1}$$

Options for first four with $n=3$

Starts 0 $\Rightarrow$ no good

Starts 10 $\Rightarrow$ 2-case with $m \mapsto m-1$

$$P(\text{good}) = \frac{m-2}{m} \qquad P(\text{case}) = \frac{m}{m+3} \cdot \frac{3}{m+2}$$

Starts 1100 $\Rightarrow$ 1-case with $m \mapsto m-2$

$$P(\text{good}) = \frac{m-2}{m-1} \qquad P(\text{case}) = \frac{m}{m+3} \cdot \frac{m-1}{m+2} \cdot \frac{3}{m+1} \cdot \frac{2}{m}$$

Starts 1101 $\Rightarrow$ good

$$P(\text{case}) = \frac{m}{m+3} \cdot \frac{m-1}{m+2} \cdot \frac{3}{m+1} \cdot \frac{m-2}{m}$$

Starts 111 $\Rightarrow$ good

$$P(\text{case}) = \frac{m}{m+3} \frac{m-1}{m+2} \frac{m-2}{m+1}$$

Total $P(\text{good}) =$

$$\frac{m-2}{\cancel{m}} \cdot \frac{\cancel{m}}{m+3} \cdot \frac{3}{m+2} + \frac{m-2}{\cancel{m-1}} \cdot \frac{\cancel{m}}{m+3} \cdot \frac{\cancel{m-1}}{m+2} \cdot \frac{3}{m+1} \cdot \frac{2}{\cancel{m}}$$

$$+ \frac{3}{\cancel{m}} \cdot \frac{\cancel{m}}{m+3} \cdot \frac{m-1}{m+2} \cdot \frac{m-2}{m+1} + \frac{m}{m+3} \cdot \frac{m-1}{m+2} \cdot \frac{m-2}{m+1}$$

$$= \frac{m-2}{m+1} \cdot \frac{1}{(m+2)(m+3)} \cdot \left(3m+3 + 6 + 3m-3 + m^2 - m\right)$$

$$= \frac{m-2}{m+1} \cdot \frac{1}{(m+2)(m+3)} \left(m^2+5m+6\right) = \frac{m-2}{m+1}$$

(13) By definition

$$1 = \int_{-a}^{0} 2k\,dx + \int_{0}^{2} k\sqrt{4-x^2}\,dx$$

$$= 2ak + k\left[2\sin^{-1}\left(\tfrac{1}{2}x\right) + \tfrac{1}{2}x\sqrt{4-x^2}\right]_0^2$$

$$= 2ak + k\pi$$

$$\therefore k = \frac{1}{2a+\pi}$$

$$\text{Mean}(X) = \int_{-a}^{0} 2kx\,dx + \int_{0}^{2} kx\sqrt{4-x^2}\,dx$$

$$= \left[kx^2\right]_{-a}^{0} + \left[-\frac{k}{3}(4-x^2)^{3/2}\right]_0^2$$

$$= -ka^2 + \frac{8k}{3}$$

$$= \frac{8-3a^2}{3(2a+\pi)}$$

By above $P(X>0) = \dfrac{\pi}{2a+\pi}$ (2nd integral)

$$P(X>0) > \frac{1}{10} \iff \frac{\pi}{2a+\pi} < \frac{1}{10} \le \quad 9\pi < 2a$$

So $2a > 9\pi \implies d < 0$

So $P(X<d) = 2k(d+a) = \dfrac{9}{10}$

$$\therefore d = \frac{9}{20k} - a = \frac{9\pi - 2a}{20}$$

(13) contd

$$d = \sqrt{2} \Rightarrow \int_{\sqrt{2}}^{2} k\sqrt{(4-x^2)}\, dx = \frac{1}{10}$$

$$\Rightarrow k\left[2\sin^{-1}\left(\frac{1}{2}x\right) + \frac{1}{2}x\sqrt{4-x^2}\right]_{\sqrt{2}}^{2} = \frac{1}{10}$$

$$\Rightarrow k\left[\pi - \frac{\pi}{2} - 1\right] = \frac{1}{10}$$

$$\Rightarrow \quad 5\pi - 10 = 2a + \pi$$

$$\Rightarrow \quad a = 2\pi - 5$$

STEP I/2011/9

The trajectory is $y = Bx - Ax^2$ for some B, A

Since $\frac{\mathrm{d}y}{\mathrm{d}x} = \tan\theta$ at $x = 0$, $B = \tan\theta$, and so

$$d_2 = \tan\theta d_1 - Ad_1^2 \qquad [1]$$

$$d_1 = \tan\theta d_2 - Ad_2^2 \qquad [2]$$

Eliminating A, $d_2^3 - \tan\theta d_1 d_2^2 = d_1^3 - \tan\theta d_1^2 d_2$

So $\tan\theta = \frac{d_1^2+d_1 d_2+d_2^2}{d_1 d_2}$

Range = $\frac{\tan\theta}{A}$

[1] and [2] $\implies A = \frac{(d_1-d_2)(\tan\theta+1)}{d_1^2-d_2^2} = \frac{\tan\theta+1}{d_1+d_2}$

$\tan\theta + 1 = \frac{(d_1+d_2)^2}{d_1 d_2}$, so $A = \frac{(d_1+d_2)}{d_1 d_2}$

Range = $\frac{\tan\theta}{A} = \frac{d_1^2+d_1 d_2+d_2^2}{d_1+d_2}$