October 2016. FP2 test paper.

(1) $z^3 = 4\sqrt{2} - 4\sqrt{2}i$

$= 8 \text{ cis}\left(-\frac{\pi}{4}\right)$

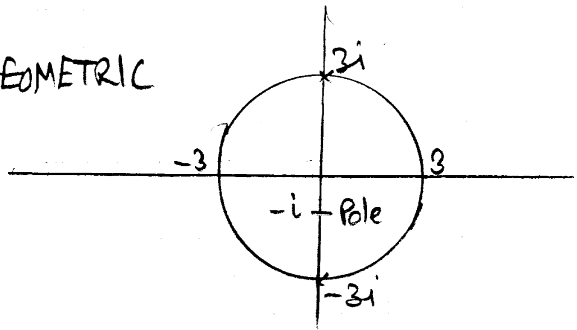
$\therefore z = 2 \text{ cis}\left(-\frac{\pi}{12}\right)$

or $2 \text{ cis } \frac{7\pi}{12}$ (adding $\frac{2\pi}{3}$)

or $2 \text{ cis}\left(\frac{-9\pi}{12}\right) = 2 \text{ cis } \frac{-3\pi}{4}$ (subtracting $\frac{2\pi}{3}$)

Check: if I try to get a 4^{th} root, it is $2 \text{cis}\left(-\frac{17\pi}{12}\right)$, which $= \text{cis } \frac{7\pi}{12}$.

(2) GEOMETRIC



By the circle inversion theorems, since the pole is not on the z-locus, the w-locus is a circle.

By symmetry, the z-diameter-ends in line with the pole map into w-diameter-ends.

$\therefore$ these are w-diameter ends

$w = \frac{3i}{3i+i} = \frac{3}{4}$

$w = \frac{-3i}{-3i+i} = \frac{3}{2}$

$\therefore$ Centre is $\frac{9}{8}$, radius is $\frac{3}{8}$

ALGEBRAIC

$$w = \frac{z}{z+i}$$

$$zw + iw = z$$

$$zw - z = iw$$

$$\therefore \quad z = \frac{iw}{w-1}$$

$$|z| = 3 \Rightarrow \left|\frac{iw}{w-1}\right| = 3$$

$$\Rightarrow |iw| = 3|w-1|$$

$$\Rightarrow |iw|^2 = 9|w-1|^2$$

$$\Rightarrow u^2 + v^2 = 9(u-1)^2 + 9v^2 \quad \text{if } w = u + iv$$

$$\Rightarrow 8u^2 - 18u + 8v^2 = -9$$

$$\Rightarrow u^2 - \frac{9}{4}u + \left(\frac{9}{8}\right)^2 + v^2 = -\frac{9}{8} + \frac{81}{64} = \frac{9}{64}$$

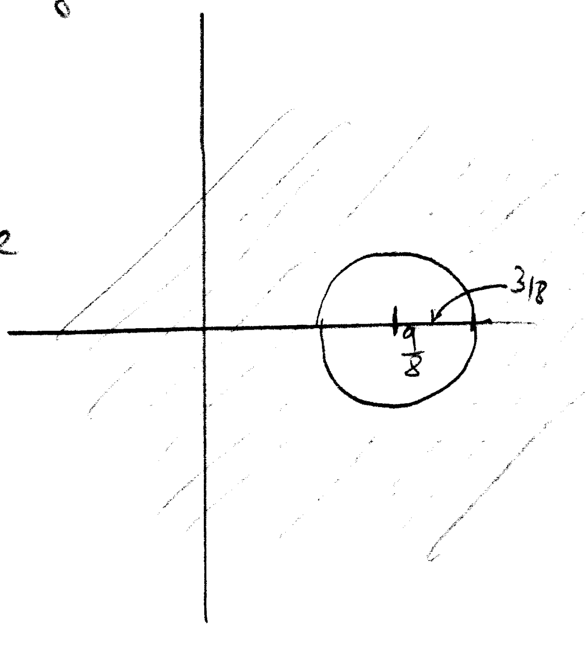
$$\Rightarrow \left(u - \frac{9}{8}\right)^2 + v^2 = \left(\frac{3}{8}\right)^2$$

$\therefore$ Centre is $\frac{9}{8}$, radius is $\frac{3}{8}$

REGION

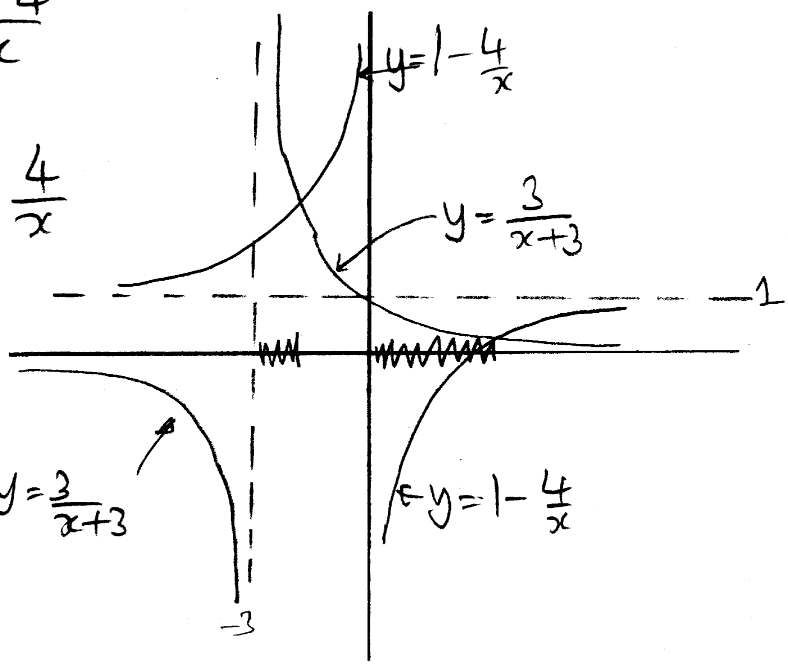
$z = 0$ maps to $w = 0$
which is outside the w-circle

$\therefore$ R is as shaded



(3) $\dfrac{3}{x+3} > \dfrac{x-4}{x}$

$\Leftrightarrow \dfrac{3}{x+3} > 1 - \dfrac{4}{x}$



Critical points where

$$3x^2(x+3) = x^2(x+3)^2 - 4x(x+3)^2$$

or $(x+3)x\left[3x - x(x+3) + 4(x+3)\right] = 0$

$x=-3$, $x=0$, or $x^2-4x-12=0$

$(x-6)(x+2)=0$

$x=-2$ or 6

From diagram, the set of values is

$-3 < x < -2$

and $0 < x < 6$

④ $(2r+1)^3 = Ar^3 + Br^2 + Cr + 1$

$$A = 8,\ B = 12,\ C = 6$$

$$\therefore\ (2r+1)^3 - (2r-1)^3 = 8r^3 + 12r^2 + 6r + 1$$
$$\qquad - 8r^3 + 12r^2 - 6r + 1$$
$$= 24r^2 + 2$$

$$\therefore\ 24n^2 + 2 = (2n+1)^3 - \cancel{(2n-1)^3}$$
$$24(n-1)^2 + 2 = \cancel{(2n-1)^3} - \cancel{(2n-3)^3}$$
$$24(n-2)^2 + 2 = \cancel{(2n-3)^3} - \cancel{(2n-5)^3}$$
$$\cdots\cdots$$
$$24 \times 2^2 + 2 = \cancel{5^3} - \cancel{3^3}$$
$$24 \times 1^2 + 2 = \cancel{3^3} - 1^3$$

Adding: $24 \sum_1^n r^2 + 2n = (2n+1)^3 - 1$

$$\therefore\ 24 \sum_1^n r^2 = (2n+1)^3 - 1 - 2n$$
$$= (2n+1)\left[(2n+1)^2 - 1\right]$$
$$= (2n+1)(2n)(2n+2) \quad \text{[difference of squares]}$$
$$= 4n(n+1)(2n+1)$$
$$\therefore\ \sum_1^n r^2 = \frac{1}{6} n(n+1)(2n+1)$$

(5) De Moivre's theorem $\Rightarrow$

$$\cos 5\theta + i\sin 5\theta = (\cos\theta + i\sin\theta)^5$$
$$= c^5 + 5ic^4s^2 - 10c^3s^2 - 10i\,c^2s^3 + 5cs^4 + is^5$$

if $c = \cos\theta$ and $s = \sin\theta$

Equate imaginary parts

$$\sin 5\theta = 5c^4s^2 - 10c^2s^3 + s^5$$
$$= s(5c^4 - 10c^2s^2 + s^4)$$
$$= s(5(1-s^2)^2 - 10(1-s^2)s^2 + s^4)$$
$$= s(16s^4 - 20s^2 + 5) \ \square$$

$$\sin 3\theta = s(3 - 4s^2)$$

$\therefore$ $5\sin 3\theta = \sin 5\theta$

$\Rightarrow s(16s^4 - 20s^2 + 5 - 15 + 20s^2) = 0$

$\Rightarrow s(16s^4 - 10) = 0$

$\Rightarrow s = 0$ or $s = 0.8891$ or $s = -0.8891$

$\Rightarrow \theta = 0, \pi, 1.095, 4.237, 5.188, 2.046$