FP1. MAY 2015

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1.

$$f(x) = 9x^3 - 33x^2 - 55x - 25$$

Given that $x = 5$ is a solution of the equation $f(x) = 0$, use an algebraic method to solve $f(x) = 0$ completely.

(5)

The equation is equivalent to

$$x^3 - \frac{11}{3}x^2 - \frac{55}{9}x - \frac{25}{9} = 0$$

Call other roots $a \pm bi$

Sum of roots $= \frac{11}{3} = 5 + a + bi + a - bi$

$\therefore a = -\frac{2}{3}$

product of roots $= \frac{25}{9} = 5\left(-\frac{2}{3} + bi\right)\left(-\frac{2}{3} - bi\right)$

$= 5\left(\frac{4}{9} + b^2\right)$

$\therefore b = \pm\frac{1}{3}$

$\therefore$ Other roots are $-\frac{2}{3} \pm \frac{1}{3}i$

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2. In the interval $13 < x < 14$, the equation

$$3 + x\sin\left(\frac{x}{4}\right) = 0, \text{ where } x \text{ is measured in radians,}$$

has exactly one root, α.

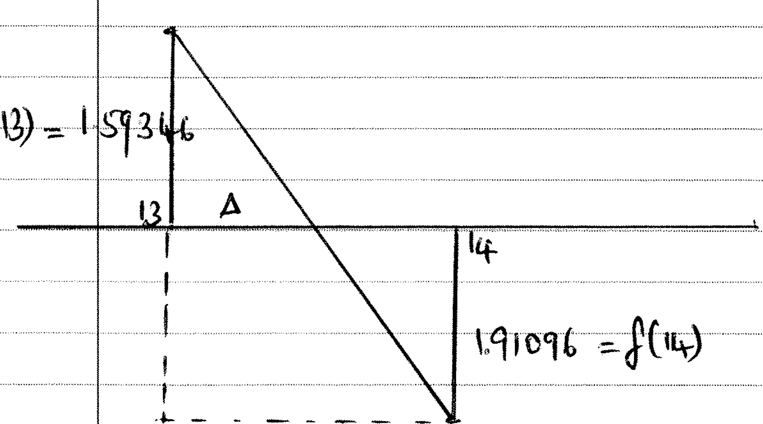
(a) Starting with the interval [13, 14], use interval bisection twice to find an interval of width 0.25 which contains α.

(3)

(b) Use linear interpolation once on the interval [13, 14] to find an approximate value for α. Give your answer to 3 decimal places.

(4)

(b)



Similar triangles $\Rightarrow \dfrac{\Delta}{1} = \dfrac{1.59346}{1.59346 + 1.91096} \Rightarrow \Delta = 0.455$

$\Rightarrow$ Next guess $= 13.455$

(a) If $f(x) = 3 + x\sin\left(\frac{x}{4}\right)$

$f(13.5) = -0.122 < 0$

$\therefore$ root is between 13 and 13.5

$f(13.25) = +0.746 > 0$

$\therefore$ root is between 13.25 and 13.5

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3. (a) Using the formulae for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$, show that

$$\sum_{r=1}^{n}(r+1)(r+4) = \frac{n}{3}(n+4)(n+5)$$

for all positive integers n.

(5)

(b) Hence show that

$$\sum_{r=n+1}^{2n}(r+1)(r+4) = \frac{n}{3}(n+1)(an+b)$$

where a and b are integers to be found.

(3)

(a) $\text{sum} = \sum_{r=1}^{n}(r^2+5r+4) = \sum_{r=1}^{n} r^2 + 5\sum_{r=1}^{n} r + \sum_{r=1}^{n} 4$

$= \frac{1}{6}n(n+1)(2n+1) + \frac{5}{2}n(n+1) + 4n$

$= \frac{n}{6}\left[2n^2+3n+1+15n+15+24\right]$

$= \frac{n}{6}\left[2n^2+18n+40\right]$

$= \frac{n}{3}[n+4][n+5]$ $\square$

(b) $\sum_{r=n+1}^{2n}(r+1)(r+4) = \sum_{1}^{2n}(r+1)(r+4) - \sum_{1}^{n}(r+1)(r+4)$

$= \frac{2n}{3}[2n+4][2n+5] - \frac{n}{3}[n+4][n+5]$

$= \frac{n}{3}\left[8n^2+36n+40-n^2-9n-20\right]$

$= \frac{n}{3}\left[7n^2+27n+20\right]$

$= \frac{n}{3}(n+1)(7n+20)$

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4.

$$z_1 = 3\text{i} \text{ and } z_2 = \frac{6}{1 + \text{i}\sqrt{3}}$$

(a) Express z_2 in the form $a + \text{i}b$, where a and b are real numbers.
(2)

(b) Find the modulus and the argument of z_2, giving the argument in radians in terms of π.
(4)

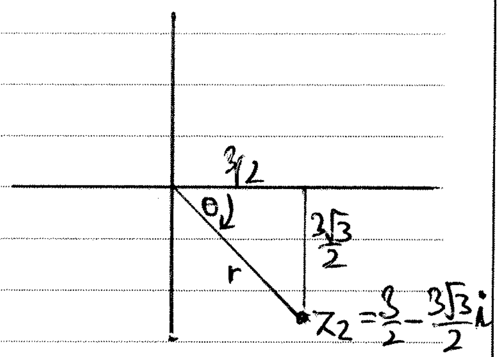
(c) Show the three points representing z_1, z_2 and $(z_1 + z_2)$ respectively, on a single Argand diagram.
(2)

(a) $z_2 = \frac{6}{1+i\sqrt{3}} \cdot \frac{1-i\sqrt{3}}{1-i\sqrt{3}} = \frac{6-6\sqrt{3}i}{4} = \frac{3}{2} - \frac{3\sqrt{3}}{2}i$

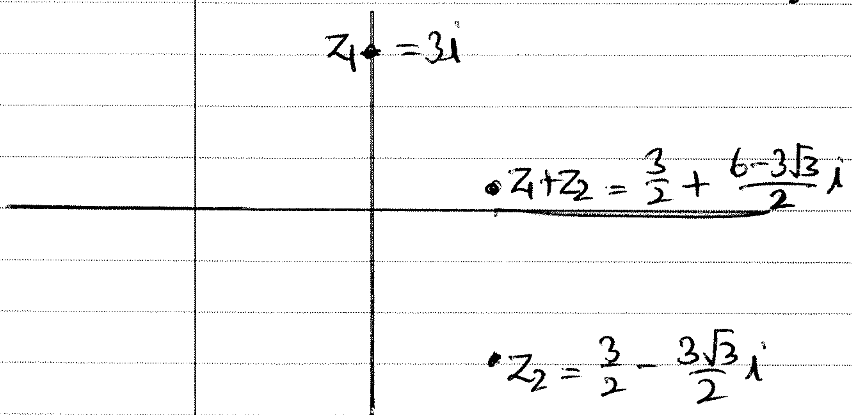
(b) $Pol\left(\frac{3}{2}, -\frac{3\sqrt{3}}{2}\right) = 3\ cis(-60°)$

$\therefore\ |z_2| = 3$

$arg(z_2) = -\frac{\pi}{3}$



(c)



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5. The rectangular hyperbola H has equation $xy = 9$

The point A on H has coordinates $\left(6, \frac{3}{2}\right)$.

(a) Show that the normal to H at the point A has equation

$$2y - 8x + 45 = 0$$

(5)

The normal at A meets H again at the point B.

(b) Find the coordinates of B.

(4)

General point on H is $\left(3t, \frac{3}{t}\right)$

$\frac{dy}{dt} = -\frac{3}{t^2}$ and $\frac{dx}{dt} = 3 \Rightarrow \frac{dy}{dx} = -\frac{1}{t^2}$

$\Rightarrow$ slope of normal $= t^2$

$\Rightarrow$ slope of normal at A, where $t = 2$, is 4

$\Rightarrow$ normal is $y = 4x + k$ for some k

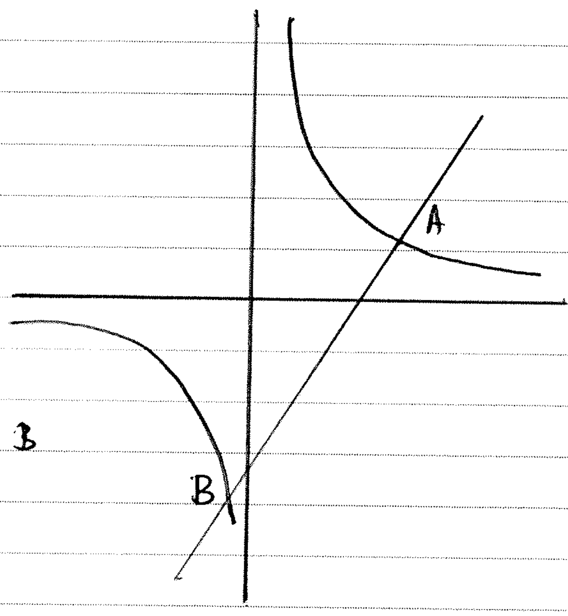
Since normal goes through A

$\frac{3}{2} = 24 + k$, and $k = -\frac{45}{2}$

$\therefore$ Equation of normal is $2y - 8x + 45 = 0$ $\square$

If B is $\left(3t, \frac{3}{t}\right)$ then

$\frac{6}{t} - 24t + 45 = 0$



$\therefore 24t^2 - 45t - 6 = 0$

$8t^2 - 15t - 2 = 0$

$(8t + 1)(t - 2) = 0$

$\therefore t = 2$ at A or $t = -\frac{1}{8}$ at B

$\therefore$ B is $\left(-\frac{3}{8}, -24\right)$

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6. (i) Prove by induction that, for $n \in \mathbb{Z}^+$,

$$\begin{pmatrix} 1 & 0 \\ -1 & 5 \end{pmatrix}^n = \begin{pmatrix} 1 & 0 \\ -\frac{1}{4}(5^n - 1) & 5^n \end{pmatrix}$$

(6)

(ii) Prove by induction that, for $n \in \mathbb{Z}^+$,

$$\sum_{r=1}^{n}(2r-1)^2 = \frac{1}{3}n\left(4n^2 - 1\right)$$

(6)

(i) Step 1: prove for $n=1$

$$\begin{pmatrix} 1 & 0 \\ -1 & 5 \end{pmatrix}^1 = \begin{pmatrix} 1 & 0 \\ -1 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{4}(5^1-1) & 5^1 \end{pmatrix} \square$$

Step 2: prove true for $n=k+1$ if true for $n=k$

True for $n=k \Rightarrow \begin{pmatrix} 1 & 0 \\ -1 & 5 \end{pmatrix}^k = \begin{pmatrix} 1 & 0 \\ -\frac{1}{4}(5^k-1) & 5^k \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 1 & 0 \\ -1 & 5 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & 0 \\ -1 & 5 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\frac{1}{4}(5^k-1) & 5^k \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ -1-\frac{1}{4}(5^{k+1}-5) & 5^{k+1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{4}(5^{k+1}-1) & 5^{k+1} \end{pmatrix} \square$$

$\therefore$ True for all n by induction.

(ii) Step 1 Prove true for $n=1$

$$(2\times1-1)^2 = 1 = \tfrac{1}{3}\times1\times(4\times1-1) \ \square$$

Step 2 prove true for $n=k+1$ if true for $n=k$

True for $n=k \Rightarrow \sum_{1}^{k}(2r-1)^2 = \frac{1}{3}k\left(4k^2-1\right)$

$$\Rightarrow \sum_{1}^{k+1}(2r-1)^2 = \tfrac{1}{3}k(4k^2-1) + (2k+1)^2$$

$$= \tfrac{1}{3}(2k+1)\left[2k^2 - k + 6k + 3\right]$$

$$= \tfrac{1}{3}(2k+1)(2k+3)(k+1)$$

$$= \tfrac{1}{3}(k+1)\left[\left[2(k+1)\right]^2 - 1\right] \square$$

$\therefore$ True for all n by induction.

7. (i)

$$\mathbf{A} = \begin{pmatrix} 5k & 3k-1 \\ -3 & k+1 \end{pmatrix}, \text{ where } k \text{ is a real constant.}$$

Given that **A** is a singular matrix, find the possible values of k.

(4)

(ii)

$$\mathbf{B} = \begin{pmatrix} 10 & 5 \\ -3 & 3 \end{pmatrix}$$

A triangle T is transformed onto a triangle T' by the transformation represented by the matrix **B**.

The vertices of triangle T' have coordinates $(0, 0)$, $(-20, 6)$ and $(10c, 6c)$, where c is a positive constant.

The area of triangle T' is 135 square units.

(a) Find the matrix $\mathbf{B}^{-1}$

(2)

(b) Find the coordinates of the vertices of the triangle T, in terms of c where necessary.

(3)

(c) Find the value of c.

(3)

(i) $\det A = 5k^2 + 5k + 9k - 3 = 5k^2 + 14k - 3$

$\det A = 0 \Rightarrow (5k-1)(k+3) = 0 \Rightarrow k = -3$ or $k = 1/5$

(ii) $\det B = 45 \therefore B^{-1} = \frac{1}{45}\begin{pmatrix} 3 & -5 \\ 3 & 10 \end{pmatrix}$

$B^{-1}T' = T = \frac{1}{45}\begin{pmatrix} 3 & -5 \\ 3 & 10 \end{pmatrix}\begin{pmatrix} 0 & -20 & 10c \\ 0 & 6 & 6c \end{pmatrix}$

$= \frac{1}{45}\begin{pmatrix} 0 & -90 & 0 \\ 0 & 0 & 90c \end{pmatrix} = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 2c \end{pmatrix}$

$\therefore$ vertices of T are $(0,0)$, $(-2,0)$ and $(0,2c)$

Area of $T = \frac{1}{2}$ base $\times$ height $= 2c$

Area of $T' = \det B \times$ area of $T = 90c = 135$

$\therefore c = 3/2$

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8. The point $P(3p^2, 6p)$ lies on the parabola with equation $y^2 = 12x$ and the point S is the focus of this parabola.

(a) Prove that $SP = 3(1 + p^2)$ **(3)**

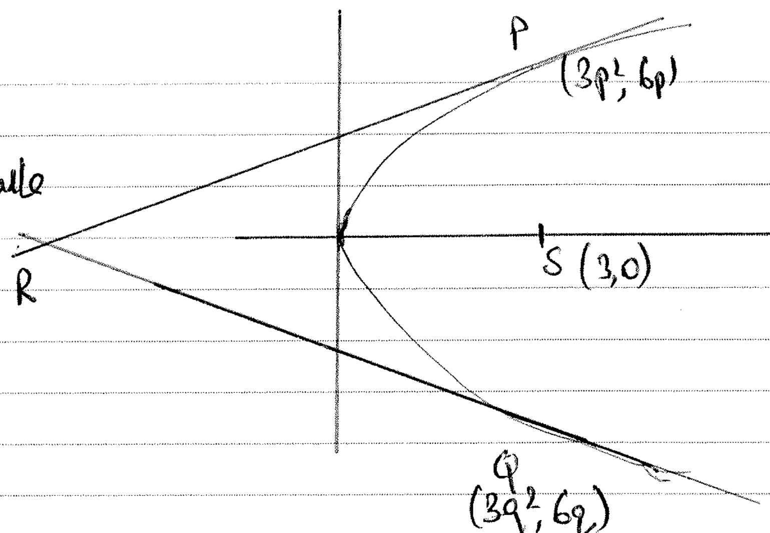
The point $Q(3q^2, 6q)$, $p \neq q$, also lies on this parabola.

The tangent to the parabola at the point P and the tangent to the parabola at the point Q meet at the point R.

(b) Find the equations of these two tangents and hence find the coordinates of the point R, giving the coordinates in their simplest form. **(8)**

(c) Prove that $SR^2 = SP.SQ$ **(3)**

We know S is (3,0) from formula book (a=3 for this parabola)



(a) By Pythagoras, $SP = \sqrt{(3p-3)^2 + (6p)^2} = \sqrt{9p^4 + 18p^2 + 9}$
$= 3(p^2+1)$ □

(b) From formula book, general point on parabola is $(3t^2, 6t)$.
At that point slope of tangent $= \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{6}{6t} = \frac{1}{t}$

$\therefore$ Tangent is $(y - 6t) = \frac{1}{t}(x - 3t^2)$, or $y = \frac{1}{t}x + 3t$
$\therefore$ R is given by solution to simultaneous eq: $\left.\begin{array}{l} y = \frac{1}{p}x + 3p \\ y = \frac{1}{q}x + 3q \end{array}\right\}$ $x = 3pq$, $y = 3(p+q)$

(c) By Pythagoras
$SR^2 = (3pq-3)^2 + [3(p+q)]^2 = 9(p^2q^2 - 2pq + 1 + p^2 + 2pq + q^2)$
$= 9(p^2+1)(q^2+1) = SP.SQ$ □